The AJ-conjecture for the Teichmüller TQFT

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Abstract

We formulate the AJ-conjecture for the Teichmüller TQFT and we prove it in the case of the figure-eight knot complement and the 5_2-knot complement. This states that the level-N Andersen-Kashaev invariant, $J_{M,K}^{(\mathrm{b},N)}$, is annihilated by the non-homogeneous $\hat{A}$-polynomial, evaluated at appropriate q-commutative operators. These are obtained via geometric quantisation on the moduli space of flat $\mathrm{SL}(2,\mathbb{C})$-connections on a genus-1 surface. The construction depends on a parameter σ in the Teichmüller space in a way measured by the Hitchin-Witten connection, and results in Hitchin-Witten covariantly constant quantum operators for the holonomy functions m and ℓ along the meridian and longitude. Their action on $J_{M,K}^{(\mathrm{b},N)}$ is then defined via a trivialisation of the Hitchin-Witten connection and the Weil-Gel'Fand-Zak transform.

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1 Overview and summary of the results

In this paper we consider the level-N Andersen-Kashaev invariant $J_{M,K}^{(\mathrm{b},N)}$, which is minor transform of the Teichmüller TQFT partition function for a knot K sitting inside a closed, oriented 3-manifold M, where b is a unitary complex quantum parameter. The level $N = 1$ Teichmüller TQFT was introduced by Andersen and Kashaev in [AK14a], and then extended to arbitrary (odd) N by Andersen and Kashaev in [AK14b] and further detailed in [AM16b]. This is a combinatorial approach to quantum $\mathrm{SL}(2,\mathbb{C})$-Chern-Simons theory, whose version for $\mathrm{SU}(2)$ has been widely studied and understood via several, different approaches over the past three decades. The theory was first considered by Witten's original paper [Wit89], where he used path-integral arguments to interpret the coloured Jones polynomial as its partition function. This was then formalised by Reshetikhin and Turaev [RT90, RT91], who incorporated the invariant in a TQFT defined by means of combinatorics. Another approach to the theory uses geometric quantisation on moduli spaces of flat connections [ADPW91, Hit90], whose result depends on a Teichmüller parameter to an extent measured by the projectively flat Hitchin connection. This viewpoint was later proved to be equivalent to the Reshetikhin-Turaev theory via a chain of isomorphisms [Las98, AU07a, AU07b, AU12, AU15] involving also conformal field theory. As for the $\mathrm{SL}(2,\mathbb{C})$ case, a formulation in terms of geometric quantisation was first proposed by Witten in [Wit91] by means of a Teichmüller space dependent real polarisation. Again, this results in a bundle of quantum Hilbert spaces, equipped with the Hitchin-Witten projectively flat connection, further studied from a purely mathematical viewpoint in a more general setting in [AG14]. Little is known, as of now, about the relation to the combinatorial viewpoint on the Teichmüller TQFT, and it would be of great interest to have an identification of these two approached just as in the case of $\mathrm{SU}(2)$. In the specific situation of a genus 1 surface, however, a specific identification between the vector spaces coming from the two different approaches is provided by the so called Weil-Gel'fand-Zak transform.

The invariant $J_{M,K}^{(\mathrm{b},N)}$ consists of a complex-valued function on $\mathbb{A}_N := \mathbb{R} \oplus \mathbb{Z}/N\mathbb{Z}$ having a natural meromorphic extension to $\mathbb{A}_N^{\mathbb{C}} := \mathbb{C} \oplus \mathbb{Z}/N\mathbb{Z}$. Its definition is based on a multiple integral involving the level-N quantum dilogarithm [AK14b], an extension of Faddeev's function to $\mathbb{A}_N$ satisfying an adaptation of the same difference equation. In [AK14a, AM16b], the invariant was conjectured to enjoy certain properties analogous to those expected for the coloured Jones polynomial, thus making it into an $\mathrm{SL}(2,\mathbb{C})$ analogous of the $\mathrm{SU}(2)$ invariant. The statement was checked for the first two hyperbolic knots by explicit computation of the invariant, using the properties of the quantum dilogarithm. The final expression was found to agree for $N = 1$ with the partition function of the quantum $\mathrm{SL}(2,\mathbb{C})$-Chern-Simons theory derived in the literature [Hik01, Hik07, Dim13, DGLZ09, DFM11]. In some of the cited works, Faddeev's equation for the quantum dilogarithm is used for showing that said partition function is annihilated by some version of the quantum $\widehat{A}$-polynomial from Garoufalidis' original AJ-conjecture [Gar04].

In the present paper we address the problem of the quantisation of the observables of the $\mathrm{SL}(2,\mathbb{C})$-Chern-Simons theory in genus one, with a specific interest for the A-polynomial of a knot. In analogy with [Gar04], we search for

q-commutative quantum operators $\widehat{m}_{\mathbf{x}}$ and $\widehat{\ell}_{\mathbf{x}}$ on functions on $\mathbb{A}_N$, associated to the holonomy functions m and ℓ on the $\mathrm{SL}(2,\mathbb{C})$-character variety of the surface. To this end, we fix a level $t = N + iS$ and run geometric quantisation on (a double cover of) the moduli space of flat connection using a real polarisation as in Witten's work [Wit91]. For every value of the Teichmüller parameter $\sigma \in \mathcal{T}$, we find that the pre-quantum operators associated to the *logarithmic* holonomy coordinates U and V preserve the polarisation. Therefore, they can be promoted to quantum operators $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$, which moreover turn out to be normal, thus admitting well-defined exponentials $\widehat{m}_\sigma$ and $\widehat{\ell}_\sigma$. Although the resulting operators depend explicitly on σ, we establish the following result.

Theorem 1. *The quantum operators $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$, and hence $\widehat{m}_\sigma$ and $\widehat{\ell}_\sigma$, are covariantly constant with respect to the Hitchin-Witten connection.*

In the case of genus one, the Hitchin-Witten connection admits an explicit trivialisation, first proposed in Witten's original work. Combining this with the Weil-Gel'Fand-Zak transform, we obtain σ-independent operators $\widehat{m}_{\mathbf{x}}$ and $\widehat{\ell}_{\mathbf{x}}$ on functions on $\mathbb{A}_N^{\mathbb{C}}$.

Theorem 2. *Let $t = N + iS$ be fixed, and put*

$$\mathrm{b} = -ie^{2rN}\,, \quad \textit{where} \quad e^{4rN} = -\frac{\overline{t}}{t}\,.$$

Then the operators $\widehat{m}$ and $\widehat{\ell}$ as above act on functions on $\mathbb{A}_N^{\mathbb{C}}$ via the Weil-Gel'fand-Zak transform as

$$\widehat{m}_{\mathbf{x}}\colon f(x,n) \mapsto e^{-2\pi\frac{\mathrm{b}x}{\sqrt{N}}} e^{2\pi i\frac{n}{N}} f(x,n)\,,$$
$$\widehat{\ell}_{\mathbf{x}}\colon f(x,n) \mapsto f\Big(x - \frac{i\mathrm{b}}{\sqrt{N}}, n+1\Big)\,.$$

Moreover, the operators make a q-commutative pair, i.e.

$$\widehat{\ell}_{\mathbf{x}}\widehat{m}_{\mathbf{x}} = q\widehat{m}_{\mathbf{x}}\widehat{\ell}_{\mathbf{x}}\,,$$

with

$$q = \exp\left(2\pi i\frac{\mathrm{b}^2+1}{N}\right) = e^{4\pi i/t}\,.$$

We then consider the algebra $\mathcal{A}$ generated by these operators, and sitting inside this the left ideal $\mathcal{I}(J_{M,K}^{(\mathrm{b},N)})$ annihilating the invariant. Following the lines of [Gar04], we define the $\widehat{A}^{\mathbb{C}}$-polynomial as a preferred element of this ideal. Using the explicit expression of the invariant for 4_1 and 5_2 inside S^3, we compute the $\widehat{A}^{\mathbb{C}}$-polynomial for these knots, and check the following conjecture for these two cases.

Conjecture 1. *Let $K \subseteq M$ be a knot inside a closed, oriented 3-manifold, with hyperbolic complement. Then the non-commutative polynomial $\widehat{A}_K^{\mathbb{C}}$ agrees with $\widehat{A}_K$ up to a right factor, linear in $\widehat{m}_{\mathbf{x}}$, and it reproduces the classical A-polynomial in the sense of the original AJ-conjecture.*

Besides its intrinsic interest in relation to the original AJ-conjecture, we also find in this statement a further indication concerning the role of the Weil-Gel'fand-Zak transform in connecting geometric quantisation and Teichmüller TQFT in genus one.

Theorem 3. *Conjecture 1 holds true for the figure-eight knot* 4_1*, and* 5_2*.*

We shall re-formulate the statements more precisely later in the discussion; the proof of the result goes as follows. For either knot, the Andersen-Kashaev invariant is a function on $\mathbb{A}_N$ defined as an integral in d$\mathbf{y}$ of an appropriate function of two variables $\mathbf{x}, \mathbf{y}$ along $\mathbb{A}_N$. The integrand is obtained as a combination of quantum dilogarithms and Gaussian functions, and it has a natural meromorphic extension to the whole $(\mathbb{A}_N^{\mathbb{C}})^2$. In order to understand the ideal, we study first the polynomials in $\widehat{m}_{\mathbf{x}}$, $\widehat{m}_{\mathbf{y}}$, $\widehat{\ell}_{\mathbf{x}}$ and $\widehat{\ell}_{\mathbf{y}}$ annihilating the integrand. We start by considering the convergence properties of the integral, so as to ensure that it can be extended to the whole $\mathbb{A}_N^{\mathbb{C}}$. For parameters a and ε in a suitable domain, we find open regions $R_{a,\varepsilon}$ on which the meromorphic extension of the invariant is given by integration along appropriate contours $\Gamma_{a,\varepsilon}$. These regions exhaust the whole $\mathbb{A}_N^{\mathbb{C}}$ as the parameters range in their domain, thus giving explicit expressions for the invariant at every given point. Next, we use Lemma 5 to express the action of $\widehat{\ell}_{\mathbf{x}}$ and $\widehat{\ell}_{\mathbf{y}}$ on the integrand in terms of $\widehat{m}_{\mathbf{x}}$ and $\widehat{m}_{\mathbf{y}}$, thus giving two operators with the desired property. With the help of a computer, we run reduction in $\widehat{m}_{\mathbf{y}}$, and since the action of $\widehat{\ell}_{\mathbf{y}}$ inside an integral is equivalent to a shift in the integration contour we can evaluate the resulting polynomial at $\widehat{\ell}_{\mathbf{y}} = 1$. Using some care, one can move the operator outside the integral and finally find an element of $\mathcal{I}_{\mathrm{loc}}(J^{(\mathrm{b},N)}_{M,K})$, which we then check to be the desired generator $\widehat{A}^{\mathbb{C}}_{t,K}$.

This paper is organised as follows. In section 2 we overview the background material we refer to throughout the the rest of the work. This includes generalities on geometric quantisation and the Hitchin-Witten connection, the level-N quantum dilogarithm and the Weil-Gel'fand-Zak transform, the Teichmüller TQFT, and the original AJ-conjecture. In section 3 we define the precise structure to which we are going to apply geometric quantisation, and argue that it provides a model for (a double cover of) the moduli space relevant for genus one Chern-Simons theory. In section 4 we actually run the geometric quantisation machinery to obtain the desired operators. First, we use the standard definition of the pre-quantum operators to quantise the logarithmic holonomy functions corresponding to the meridian and longitude on the torus. Next, we check that the operators are compatible with the chosen polarisation, thus descending to *quantum* operators which are Hitchin-Witten covariantly constant. We then show that the operators can be consistently exponentiated, whence we define quantum operators for the exponential holonomy eigenvalues, which in fact generate the algebra of regular functions on the character variety. Finally, we present the explicit trivialisation of the Hitchin-Witten connection and use it to make the operators independent on the Teichmüller parameter. We conclude the section by determining the action of the operators on functions on $\mathbb{A}_N$ via by the Weil-Gel'fand-Zak transform. In section 5, we explicitly carry out the procedure described above to determine the $\widehat{A}^{\mathbb{C}}$-polynomial for the first two hyperbolic knots.

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2 Introduction

2.1 Geometric quantisation

By a pre-quantum line bundle on a symplectic manifold (M, ω) one means a triple $(\mathscr{L}, h, \nabla)$ consisting of a complex line bundle on M, a Hermitian structure and a compatible connection with curvature $-i\omega$. Given this data, geometric pre-quantisation defines a pre-quantum Hilbert space consisting of the square-summable sections of the bundle, and for every smooth real function f on M a pre-quantum operator

$$\widehat{f} = f - i\nabla_{H_f} , \tag{1}$$

where f is identified with the multiplication by f, and H_f denotes its Hamiltonian vector field. This definition gives self-adjoint operators on the pre-quantum Hilbert space and ensures the Dirac quantisation condition for the operators associated to functions f and g

$$[\widehat{f}, \widehat{g}] = -i\widehat{\{f, g\}} .$$

The association of $\widehat{f}$ to f is $\mathbb{R}$-linear, and one may extend it to complex-valued functions by $\mathbb{C}$-linearity thus (1) remains valid for complex f, but in this case, however, $\widehat{f}$ need not be self-adjoint.

The full geometric quantisation also requires a polarisation on M, i.e. an involutive Lagrangian distribution P in the complexified tangent bundle, such that $\dim(P \cap \overline{P})$ is constant. Unlike for pre-quantisation, the exact construction of the quantum Hilbert space may depend on the specific situation and the kind of polarisation at hand. The core idea, however, is that the quantum states should correspond to polarised sections of the pre-quantum line bundle, i.e. those which are covariantly constant along P, or an adaptation thereof. It is important to stress that, in general, the pre-quantum operators do *not* automatically descend to quantum operators on this space.

When $P \subseteq \mathrm{T}\, M$ is real, involutivity means that P is a Lagrangian foliation. When this is the case, there exists a natural linear connection on each leaf, which allows one to make sense of geodesics on the leaves, and hence geodesic completeness. The connection turns out to be flat, and endows every simply connected and complete leaf with the structure of an affine space. The restriction of a function f on M is affine linear on every leaf if and only if the Poisson bracket $\{f, g\}$ is constant along P whenever g is. As it turns out, the pre-quantum operator $\widehat{f}$ preserves the space of polarised sections of a pre-quantum line bundle if and only if f is affine linear on the leaves.

We shall consider the situation when P is real with simply connected and complete leaves which intersect a given compact, symplectic subspace $M_{\mathrm{red}} \subseteq M$ at exactly one point. In this case, parallel transport along the leaves defines a bijection between smooth polarised sections of $\mathscr{L}$ over M and all smooth sections over M_{red}. Although the L^2 product of polarised sections on M diverges in general, the pairing of their restrictions to M_{red} can always be performed.

Using this as the inner product and taking completion defines the quantum Hilbert

$$\mathcal{H}_P \simeq \mathrm{L}^2\big(M_{\mathrm{red}}, \mathscr{L}|_{M_{\mathrm{red}}}\big)\,.$$

We denote this as an isomorphism rather than an equality to stress the dependence on P, which would otherwise disappear once restricting to M_{red}.

The setting described above is the one in Witten's quantisation of Chern-Simons theory on a Riemann surface Σ for a complex group $G_{\mathbb{C}}$ in [Wit91], which is our motivational example and main interest. The construction uses as M and M_{red} the moduli spaces $\mathcal{M}_{\mathbb{C}}$ and $\mathcal{M}$ of flat $G_{\mathbb{C}}$- and G-connections, where $G \subseteq G_{\mathbb{C}}$ is a compact real form, and the line bundle $\mathscr{L}$ is the Chern-Simons pre-quantum line bundle. We will give some of the details concerning the construction for the case of a surface of genus 1 in paragraph 3.1. For the time being, we only wish to stress that the data of the Riemann surface structure σ on Σ is extrinsic for Chern-Simons theory. The choice of σ enters the picture through the polarisation, and although all the resulting spaces $\mathcal{H}_\sigma$ are explicitly isomorphic to a common model, this identification is in fact of no physical relevance. For this reason, Witten arranges the quantum Hilbert spaces to form an infinite-rank vector bundle over the Teichmüller space and defines the appropriate connection for measuring the dependence on σ. This, called the Hitchin-Witten connection, is based on the Kähler structure J induced by σ on $\mathcal{M}$, and once defined it allows to remove $\mathcal{M}_{\mathbb{C}}$ from the picture. As was argued in Witten's work, the most remarkable feature of the Hitchin-Witten connection is that it is projectively flat. Since the Teichmüller space is contractible, this allows us to identify the quantum Hilbert spaces coming from different values of σ via holonomy, up to projective factors as the only ambiguity.

The Hitchin-Witten connection was considered and studied in a more general and abstract context in [AG14], whose main definitions and constructions we shall now summarise. Suppose that, as above, (M, ω) is a compact symplectic manifold equipped with a pre-quantum line bundle $(\mathscr{L}, h, \nabla)$, and assume that there exists an integer λ such that

$$c_1(M, \omega) = \lambda\Big[\frac{\omega}{2\pi}\Big] \in H^2(M, \mathbb{Z})\,. \tag{2}$$

Let $\mathcal{T}$ be a complex manifold, J_σ a Kähler structure on (M, ω) for every σ, in the form of an almost complex structure. With J come operators ∂_σ and $\overline{\partial}$ and a family of metrics $g_\sigma = \omega \cdot J_\sigma$, with their Levi-Civita connections, Riemann and Ricci curvature tensors and Ricci form ρ_σ. As a general property of Kähler manifolds, the first Chern class of (M, ω) is represented up to a factor 2π by the Ricci form, so one the above condition means that $\rho_\sigma - \lambda\omega$ is a real, exact 2-form. By the global $i\partial\overline{\partial}$-lemma, there exists a real function F_σ such that

$$2i\partial_\sigma\overline{\partial}_\sigma F_\sigma = \rho_\sigma - \lambda\omega\,.$$

With the constraint that it should average to 0, the function F_σ is uniquely determined, and it is called the Ricci potential of the Kähler structure.

For every point $p \in M$, the tensor $J_\sigma(p)$ takes value in the finite-dimensional vector space $\mathrm{End}(\mathrm{T}_p M)$, so one can make sense of point-wise differentiability of J in σ. If J is differentiable along every vector V tangent to $\mathcal{T}$, and if $V[J]_\sigma$ is a smooth tensor field, then J is called a smooth family of Kähler structures parametrised by $\mathcal{T}$. When this is the case, one may think of the variation of J

as a 1-form on $\mathcal{T}$ taking values in tensor fields on M, and it is customary to define a form $\tilde{G}$ by

$$V[J] = \tilde{G}(V)\cdot\omega$$

If $g = \omega\cdot J$ is the metric associated to J, and $\tilde{g}$ its inverse bi-vector field, then $\tilde{G}$ is the opposite of the variation of $\tilde{g}$. It can be checked that the $(1,1)$ part of $\tilde{G}(V)$ vanishes for every vector V tangent to $\mathcal{T}$, so one has a decomposition

$$\tilde{G}(V) = G(V) + \overline{G}(V)$$

with $G(V)$ of type $(2,0)$ and $\overline{G}(V)$ of type $(0,2)$, and both bi-vectors are symmetric. On the other hand, $\tilde{G}$ may also be decomposed into its $(1,0)$ and $(0,1)$ parts as a form on $\mathcal{T}$. When these two decompositions agree, J is called holomorphic, and rigid if for every V the tensor field $G(V)$ is holomorphic.

Suppose now that B is any (possibly complex) symmetric bi-vector field on M, and let ψ be a smooth section of $\mathscr{L}$. Using the connection on the bundle, one may consider the covariant differential of ψ as an $\mathscr{L}$-valued 1-form on M, and contract it with B. The result is a smooth section of $\mathrm{T}\,M\otimes\mathscr{L}$, which may then be differentiated again using the Levi-Civita connection on the first factor. We define $\Delta_B\psi$ to be the result of the contraction of the two indices of this object. In short, this can be written in either of the alternative forms

$$\Delta_B\psi := \nabla\cdot\big(B\cdot\nabla\psi\big) = \nabla_\mu B^{\mu\nu}\nabla_\nu\psi = B^{\mu\nu}\nabla_\mu\nabla_\nu\psi - (\delta B)^\mu\nabla_\mu\psi\,.$$

As a differential operator, Δ_B may be thought of as a Laplace operator with coefficients given by B. For instance, it can be checked that for Δ the usual Laplace operator one has

$$\Delta_{\tilde{g}} = \Delta\,, \qquad \Delta_{\tilde{G}(V)} = -V[\Delta]\,.$$

Definition 1. *Let (M,ω) be a symplectic manifold with pre-quantum line bundle $(\mathscr{L},h,\nabla)$, and J a holomorphic and rigid family of Kähler structures on it parametrised by a complex manifold $\mathcal{T}$. For each $t = k + is$ with $k\in\mathbb{Z}_{>0}$ and $s\in\mathbb{R}$, we denote by $\mathcal{H}^{(t)}$ the trivial bundle of Hilbert spaces over $\mathcal{T}$*

$$\mathcal{H}^{(t)} := \mathcal{T}\times\mathrm{L}^2\big(M,\mathscr{L}^{\otimes k}\big)\,.$$

Consider the $\mathrm{End}(\mathcal{H})$*-valued 1-forms on $\mathcal{T}$*

$$\begin{aligned} b(V) &:= \Delta_{G(V)} + 2\nabla_{G(V)\cdot\mathrm{d}F} - 2\lambda V'[F]\,,\\ \overline{b}(V) &:= \Delta_{\overline{G}(V)} + 2\nabla_{\overline{G}(V)\cdot\mathrm{d}F} - 2\lambda V''[F]\,. \end{aligned}$$

Denoting the trivial connection on $\mathcal{H}$ by ∇^{Tr}, the level t Hitchin-Witten connection is defined as

$$\tilde{\boldsymbol{\nabla}}_V = \nabla^{Tr}_V + \frac{1}{2t}b(V) - \frac{1}{2\bar{t}}\overline{b}(V) + V[F]\,.$$

In the setting of [Wit91], this agrees indeed with the connection defined there. One of the main results of [AG14] states that the Hitchin-Witten connection is projectively flat provided that $b_1(M) = 0$ and that the bi-holomorphism group of (M,J_σ) is discrete for every σ. These hypotheses are verified for the moduli spaces considered by Witten when the underlying surface has genus $g\geq 2$. Although in genus 1 they are violated, it is not hard to check directly that in that case the connection is in fact flat. This is the case we are going to study.

2.2 $\mathbb{A}_N$ and the level-N quantum dilogarithm

Definition 2. *For every positive integer N, let $\mathbb{A}_N$ be the locally compact Abelian group $\mathbb{R} \oplus \mathbb{Z}/N\mathbb{Z}$ endowed with the nomalised Haar measure $\mathrm{d}(x,n)$ defined by*

$$\int_{\mathbb{A}_N} f(x,n)\,\mathrm{d}(x,n) := \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \int_{\mathbb{R}} f(x,n)\ \mathrm{d}x\,. \tag{3}$$

We denote by $\mathcal{S}(\mathbb{A}_N, \mathbb{C})$ the space of Schwartz class functions on $\mathbb{A}_N$, i.e. functions $f(x,n)$ on $\mathbb{A}_N$ which restrict to Schwartz class functions on $\mathbb{R}$ for every n. We shall denote $\mathbb{C} \oplus \mathbb{Z}/N\mathbb{Z}$ by $\mathbb{A}_N^{\mathbb{C}}$.

Of course $\mathcal{S}(\mathbb{A}_N, \mathbb{C})$ sits inside the space $\mathrm{L}^2(\mathbb{A}_N, \mathbb{C})$ of square-summable functions, as a dense subspace. We shall often refer to a pair in $\mathbb{A}_N$ with the first entry in bold, e.g. $\mathbf{x} = (x,n)$; moreover, if $\lambda \in \mathbb{C}$ we shall write $\mathbf{x} + \lambda$ as a short-hand for $(x+\lambda, n)$.

As in [AK14b], we use the following notation for Fourier Kernels on $\mathbb{A}_N$

$$\big\langle (x,n), (y,m) \big\rangle = e^{2\pi i x y} e^{-2\pi i n m/N}\,.$$

It is straightforward to check that

$$\langle -\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle^{-1} = \langle \mathbf{x}, -\mathbf{y} \rangle\ , \qquad \langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle\,.$$

Also, the Gaussian function on $\mathbb{A}_N$ is denoted by

$$\big\langle (x,n) \big\rangle = e^{\pi i x^2} e^{-\pi i n(n+N)/N}\,.$$

Clearly, the Gaussian is symmetric, i.e. $\langle \mathbf{x} \rangle = \langle -\mathbf{x} \rangle$.

Fix now b, a complex unitary parameter with $\mathrm{Re}(\mathrm{b}) > 0$ and $\mathrm{Im}(\mathrm{b}) \geq 0$, and introduce constants c_{b} and q defined by

$$c_{\mathrm{b}} := \frac{i(\mathrm{b}+\mathrm{b}^{-1})}{2} = i\,\mathrm{Re}(\mathrm{b})\,, \qquad q^{\frac{1}{2}} := -e^{\pi i \frac{\mathrm{b}^2+1}{N}} = \Big\langle \Big(\frac{i\mathrm{b}}{\sqrt{N}}, -1 \Big) \Big\rangle^{-1}\,.$$

We shall summarise here the fundamental properties of the level N quantum dilogarithm which are relevant for this work. For the precise definition and further details see for example [AK14b, AM16b]. For N a positive *odd* integer, the quantum dilogarithm D_{b} at level N and quantum parameter b is a meromorphic function defined on $\mathbb{A}_N^{\mathbb{C}}$. One of its fundamental properties is that it solves the Faddeev difference equations

$$\mathrm{D}_{\mathrm{b}}\Big(x \pm \frac{i\mathrm{b}}{\sqrt{N}}, n \pm 1 \Big) = \Big(1 - e^{\pm \frac{\mathrm{b}^2+1}{N}} e^{2\pi \frac{\mathrm{b}}{\sqrt{N}} x} e^{2\pi i \frac{n}{N}} \Big)^{\mp 1} \mathrm{D}_{\mathrm{b}}(x,n)\,,$$

$$\mathrm{D}_{\mathrm{b}}\Big(x \pm \frac{i\overline{\mathrm{b}}}{\sqrt{N}}, n \mp 1 \Big) = \Big(1 - e^{\pm \frac{\overline{\mathrm{b}}^2+1}{N}} e^{2\pi \frac{\overline{\mathrm{b}}}{\sqrt{N}} x} e^{-2\pi i \frac{n}{N}} \Big)^{\mp 1} \mathrm{D}_{\mathrm{b}}(x,n)\,.$$

It also satisfies the inversion relation

$$\mathrm{D}_{\mathrm{b}}(\mathbf{x}) \mathrm{D}_{\mathrm{b}}(-\mathbf{x}) = \zeta_{N,\mathrm{inv}}^{-1} \langle \mathbf{x} \rangle\ , \qquad \zeta_{N,\mathrm{inv}} = e^{\pi i (N + 2c_{\mathrm{b}}^2 N^{-1})/6}\,.$$

The zeroes of this function are located at the points

$$\Big(-\frac{c_{\mathrm{b}} + i\alpha\mathrm{b} + i\beta\overline{\mathrm{b}}}{\sqrt{N}}\,,\, \beta - \alpha \Big) \qquad \text{for } \alpha, \beta \in \mathbb{Z}_{\geq 0}\,.$$

By the inversion relation, the poles occur at the opposites of these points.

The following lemma is particularly relevant for studying the convergence of integrals involving the quantum dilogarithm.

Lemma 4. *For $n \in \mathbb{Z}/N\mathbb{Z}$ fixed, the quantum dilogarithm has the following asymptotic behaviour for $x \to \infty$:*

$$\mathrm{D}_{\mathrm{b}}(x,n) \approx \begin{cases} 1 & \text{on } \left|\arg(x)\right| > \frac{\pi}{2} + \arg(\mathrm{b}), \\ \zeta_{N,\mathrm{inv}}^{-1} \langle \mathbf{x} \rangle & \text{on } \left|\arg(x)\right| < \frac{\pi}{2} - \arg(\mathrm{b}). \end{cases}$$

Furthermore, the dilogarithm enjoys the following unitarity property

$$\overline{\mathrm{D}_{\mathrm{b}}(x,n)}\mathrm{D}_{\mathrm{b}}(\overline{x},n) = 1\,.$$

It is convenient for the following discussion to change the notation according to [AM16b]. We shall call

$$\varphi_{\mathrm{b}}(x,n) := \mathrm{D}_{\mathrm{b}}(x,-n)\,.$$

The zeroes and poles of φ_{b} occur at the points $\mathbf{p}_{\alpha,\beta}$ and $-\mathbf{p}_{\alpha,\beta}$ respectively, for $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, where

$$\mathbf{p}_{\alpha,\beta} := \Big(-\frac{c_{\mathrm{b}} + i\alpha\mathrm{b} + i\beta\overline{\mathrm{b}}}{\sqrt{N}}\,,\, \alpha - \beta\Big)$$

We shall often use T to refer to the infinite closed triangle

$$T = \left\{ x \in \mathbb{C} : x \text{ lies below } -\frac{c_{\mathrm{b}}}{\sqrt{N}} + i\mathbb{R}\mathrm{b} \text{ and } -\frac{c_{\mathrm{b}}}{\sqrt{N}} + i\mathbb{R}\overline{\mathrm{b}} \right\}.$$

In particular, the zeroes and poles of $\varphi_{\mathrm{b}}(x,n)$ for n fixed occur only for $x \in T$ and $x \in -T$ respectively. Lemma 4 holds unchanged for φ_{b} in place of D_{b}.

Definition 3. *If $k \in \mathbb{Z}_{>0}$ and $\boldsymbol{\mu} : (\mathbb{A}_N^{\mathbb{C}})^k \to \mathbb{A}_N^{\mathbb{C}}$ is a $\mathbb{Z}$-linear function, let us denote by $\widehat{m}_{\boldsymbol{\mu}}$ the operator acting on complex-valued functions on $(\mathbb{A}_N^{\mathbb{C}})^k$ as*

$$\widehat{m}_{\boldsymbol{\mu}} f := \Big\langle \boldsymbol{\mu}, \Big(\frac{i\mathrm{b}}{\sqrt{N}}, -1\Big) \Big\rangle f\,.$$

Moreover, $\widehat{\ell}_{\mathbf{x}}$ is the following operator acting on $\mathbb{C}$-valued functions on $\mathbb{A}_N^{\mathbb{C}}$ by

$$\widehat{\ell}_{\mathbf{x}} f(x,n) := f\Big(x - \frac{i\mathrm{b}}{\sqrt{N}}, n+1\Big)\,.$$

It is clear that the operator $\widehat{m}_{\mu}$ always restricts to functions defined on $\mathbb{A}_N^k$, and that in particular for $\boldsymbol{\mu} = \mathbf{x}$ one gets explicitly

$$\widehat{m}_{\mathbf{x}} f(\mathbf{x}) = e^{-2\pi \frac{\mathrm{b}x}{\sqrt{N}}} e^{2\pi i \frac{n}{N}} f(\mathbf{x})$$

On the other hand, the condition for b to have positive real part implies that $i\mathrm{b}$ is never real, so strictly speaking $\widehat{\ell}_{\mathbf{x}}$ is only defined for functions on $\mathbb{A}_N^{\mathbb{C}}$. However, every analytic function $f \colon \mathbb{A}_N \to \mathbb{C}$ with infinite radius of convergence has a unique holomorphic extension to $\mathbb{A}_N^{\mathbb{C}}$. One can then make sense of the action of

$\widehat{\ell}_{\mathbf{x}}$ by applying the shift to the extended function and then restricting back to $\mathbb{A}_N$. The set of square-summable such functions is dense in $\mathrm{L}^2(\mathbb{A}_N)$, so $\widehat{\ell}_{\mathbf{x}}$ is a densely defined operator on this space.

The following lemma is an immediate consequence of the definitions and Faddeev's difference equation.

Lemma 5. *The operator $\widehat{\ell}_{\mathbf{x}}$ acts on the Gaussian as*

$$\widehat{\ell}_{\mathbf{x}} \langle \mathbf{x} \rangle = q^{-\frac{1}{2}} \widehat{m}_{\mathbf{x}}^{-1} \langle \mathbf{x} \rangle \ ,$$

and on the quantum dilogarithm as

$$\widehat{\ell}_{\mathbf{x}} \varphi_{\mathrm{b}}(\mathbf{x}) = \left(1 + q^{-\frac{1}{2}} \widehat{m}_{\mathbf{x}}^{-1}\right) \varphi_{\mathrm{b}}(\mathbf{x}) \,,$$
$$\widehat{\ell}_{\mathbf{x}}^{-1} \varphi_{\mathrm{b}}(\mathbf{x}) = \left(1 + q^{\frac{1}{2}} \widehat{m}_{\mathbf{x}}^{-1}\right)^{-1} \varphi_{\mathrm{b}}(\mathbf{x}) \,.$$

Moreover, the following commutation relation holds

$$\widehat{\ell}_{\mathbf{x}} \widehat{m}_{\mathbf{x}} = q \widehat{m}_{\mathbf{x}} \widehat{\ell}_{\mathbf{x}} \,.$$

2.3 Garoufalidis's original AJ-conjceture

Following the notation of [Gar04], we consider the q-commutative algebra

$$\mathcal{A} := \mathbb{Z}[q^{\pm 1}] \langle Q, E \rangle \Big/ \left(EQ - qQE\right) ,$$

One can also make sense of inverting polynomials in Q, and embed the above algebra into

$$\mathcal{A}_{\mathrm{loc}} := \left\{ \sum_{k=0}^{l} a_k(q,Q) E^k \colon l \in \mathbb{Z}_{\geq 0},\ a_k \in \mathbb{Q}(q,Q) \right\} \tag{4}$$

with product given by

$$\left(a(q,Q)E^k\right) \cdot \left(b(q,Q)E^h\right) := a(q,Q) b(q, q^k Q) E^{k+h} \,.$$

In the above mentioned work, Garoufalidis considers the space

$$\mathcal{F} = \left\{ \, f : \mathbb{N} \to \mathbb{Q}(q) \, \right\}$$

and the action of $\mathcal{A}$ and $\mathcal{A}_{\mathrm{loc}}$ determined by

$$Qf(n) = q^n f(n) \,, \qquad Ef(n) = f(n+1) \,.$$

It is immediately checked that this defines indeed an algebra representation. For a given element $f \in \mathcal{F}$ one can consider the sets of annihilators of f, which are in fact left ideals:

$$\mathcal{I}_{\mathrm{loc}}(f) = \left\{ \, p(Q,E) \in \mathcal{A}_{\mathrm{loc}} \colon p(Q,E) f = 0 \, \right\} \ , \qquad \mathcal{I}(f) = \mathcal{I}_{\mathrm{loc}}(f) \cap \mathcal{A} \,.$$

Since every ideal in $\mathcal{A}_{\mathrm{loc}}$ is principal, such a function defines uniquely a generator $\widehat{A}_{q,f}$ of $\mathcal{I}_{\mathrm{loc}}(f)$ of minimal degree in E and co-prime coefficients in $\mathbb{Z}[q,Q]$. The AJ conjecture, as formulated in [Gar04], states then the following.

Conjecture 2 (AJ Conjecture)**.** *Let K be a knot in S^3 and $J_K : \mathbb{N} \to \mathbb{Z}[q^{\pm 1}]$ its coloured Jones function, and abbreviate $\widehat{A}_{q,J_K}$ as $\widehat{A}_{q,K}$. Then, up to a left factor, $\widehat{A}_{q,K}(Q,E)$ returns the classical A-polynomial of K when evaluated at $q = 1$, $Q = m^2$ and $E = \ell$.*

Given $f \in \mathcal{F}$ one may also study the problem of finding $p \in \mathcal{A}$ such that

$$p(Q,E) \in \mathbb{Q}(q^n, q)\,,$$

and define the non-homogeneous $\widehat{A}_q^{\mathrm{nh}}$-polynomial of f as the minimal degree solution with co-prime coefficients in $\mathbb{Z}[q,Q]$. Clearly, if $\widehat{A}_q^{\mathrm{nh}}(f)f = B(q^n,q)$, then one can divide both sides on the left by $B(Q,q)$ in $\mathcal{A}_{\mathrm{loc}}$ to find an operator sending f to a constant, so

$$(E-1)\left(\frac{1}{B(Q,q)}\widehat{A}_q^{\mathrm{nh}}(f)\right)f = 0\,.$$

This can be used to retrieve the $\widehat{A}_q$-polynomial of f and state an AJ-conjecture for the non-homogeneous $\widehat{A}_q^{\mathrm{nh}}$-polynomial.

The problem of the non-homogeneous recursion is addressed in [GS10] for the specific class of twist knots K_p. The figure-eight knot 4_1 and 5_2 correspond to K_p for $p = -1$ and $p = 2$, respectively, and in the cited work the homogeneous $\widehat{A}_q^{\mathrm{nh}}$-polynomials for these knots are explicitly found to be

$$\begin{aligned}\widehat{A}_{q,4_1}^{\mathrm{nh}} &= q^2Q^2\big(q^2Q-1\big)\big(qQ^2-1\big)E^2\\ &\quad-\big(qQ-1\big)\Big(q^4Q^4-q^3Q^3-q(q^2+1)Q^2-qQ+1\Big)E\\ &\quad+q^2Q^2\big(Q-1\big)\big(q^3Q^2-1\big)\,,\end{aligned} \tag{5}$$

$$\begin{aligned}\widehat{A}_{q,5_2}^{\mathrm{nh}} = &\big(q^3Q-1\big)\big(qQ^2-1\big)\big(q^2Q^2-1\big)E^3\\ &+q\big(q^2Q-1\big)\big(qQ^2-1\big)\big(q^4Q^2-1\big)\\ &\cdot\Big(q^9Q^5-q^7Q^4-q^4(q^3-q^2-q+1)Q^3+q^2(q^3+1)Q^2+2q^2Q-1\Big)E^2\\ &-q^5Q^2\big(qQ-1\big)\big(q^2Q^2-1\big)\big(q^5Q^2-1\big)\\ &\cdot\Big(q^6Q^5-2q^5Q^4-q^2(q^3+1)Q^3+q(q^3-q^2-q+1)Q^2+qQ-1\Big)E\\ &+q^9Q^7\big(Q-1\big)\big(q^4Q^2-1\big)\big(q^5Q^2-1\big)\,.\end{aligned} \tag{6}$$

Notice how a factor (q^jQ-1) appears next to each E^j; by the commutation relation, each of these can be turned into $(Q-1)$ and taken to the right. These non-commutative polynomials can be explicitly compared to their classical counterparts

$$A_{4_1}(m,\ell) = m^4\ell^2 - \Big(m^8-m^6-2m^4-m^2+1\Big)\ell + m^4\,, \tag{7}$$

$$\begin{aligned}A_{5_2}(m,\ell) = \ell^3 &+ \Big(m^{10}-m^8+2m^4+2m^2-1\Big)\ell^2\\ &- m^4\Big(m^{10}-2m^8-2m^6+m^2-1\Big)\ell + m^{14}\,.\end{aligned} \tag{8}$$

These polynomials are known to be irreducible [HS04].

2.4 The Andersen-Kashaev theory

The Andersen-Kashaev theory defines an infinite-rank TQFT $\mathcal{Z}$ from quantum Teichmüller theory, which in particular defines an invariant $\mathcal{Z}(M,K)$ for every hyperbolic knot K inside a closed, oriented 3-manifold M. The following conjecture was stated first in [AK14a], and then generalised to the present form in [AM16b].

Conjecture 3 ([AK14a, AM16b])**.** *Let M be a closed oriented compact 3-manifold. For any hyperbolic knot $K \subseteq M$, there exists a two-parameter (b, N) family of smooth functions $J_{M,K}^{(\mathrm{b},N)}(\mathbf{x})$ on $\mathbb{A}_N = \mathbb{R} \times \mathbb{Z}/N\mathbb{Z}$ which enjoys the following properties:*

1. *For any fully balanced shaped ideal triangulation X of the complement of K in M, there exist a gauge-invariant real linear combination of dihedral angles λ, and a (gauge-dependent) real quadratic polynomial of dihedral angles ϕ, such that*
$$\mathcal{Z}_{\mathrm{b}}^{(N)}(X) = e^{ic_{\mathrm{b}}^2\phi} \int_{\mathbb{A}_N} J_{M,K}^{(\mathrm{b},N)}(\mathbf{x}) e^{i\lambda c_{\mathrm{b}} x}\, \mathrm{d}\mathbf{x}\,. \tag{9}$$
2. *For any vertex shaped H-triangulation Y of the pair (M,K), there exists a real quadratic polynomial of dihedral angles φ such that*
$$\lim_{\omega_Y \to \tau} \mathrm{D_b}\left(c_{\mathrm{b}} \frac{\omega_Y(K) - \pi}{\pi\sqrt{N}}, 0\right) \mathcal{Z}_{\mathrm{b}}^{(N)}(Y) = e^{ic_{\mathrm{b}}^2\phi - i\frac{\pi N}{12}} J_{M,K}^{(\mathrm{b},N)}(0,0)\,. \tag{10}$$
3. *The hyperbolic volume of the complement of K in M is recovered as the limit*
$$\lim_{\mathrm{b}\to 0} 2\pi \mathrm{b}^2 N \log\left| J_{M,K}^{(\mathrm{b},N)}(0,0)\right| = -\operatorname{Vol}(M \setminus K)\,. \tag{11}$$

In the same works, the conjecture is proven for the knots 4_1 and 5_2 in S^3, in which cases $J_{M,K}^{(\mathrm{b},N)}$ is found to be

$$J_{S^3,4_1}^{(\mathrm{b},N)}(\mathbf{x}) = e^{4\pi i \frac{c_{\mathrm{b}} x}{\sqrt{N}}} \chi_{4_1}(\mathbf{x})\,, \qquad \chi_{4_1}(\mathbf{x}) = \int_{\mathbb{A}_N} \frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y}) \langle \mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y}) \langle \mathbf{y}\rangle^{-2}}\, \mathrm{d}\mathbf{y}\,,$$

$$J_{S^3,5_2}^{(\mathrm{b},N)}(\mathbf{x}) = e^{2\pi i \frac{c_{\mathrm{b}} x}{\sqrt{N}}} \chi_{5_2}(\mathbf{x})\,, \qquad \chi_{5_2}(\mathbf{x}) = \int_{\mathbb{A}_N} \frac{\langle \mathbf{y}\rangle \, \langle \mathbf{x}\rangle^{-1}}{\varphi_{\mathrm{b}}(\mathbf{y}+\mathbf{x}) \varphi_{\mathrm{b}}(\mathbf{y}) \varphi_{\mathrm{b}}(\mathbf{y}-\mathbf{x})}\, \mathrm{d}\mathbf{y}\,.$$

Remark 6. *The expressions above differ from those in [AK14a, AM16b] by an exponential factor. Because λ is a linear combination of the dihedral angles, which are constrained by a linear, non-homogeneous relation, this extra factor can be re-absorbed into it. As is easily checked, this does not affect the asymptotic properties of the invariant, nor the validity of Conjecture 3. The role of this factor will be discussed further later on.*

In the case of level $N = 1$, these expressions agree with those found in the literature for the partition functions of Chern-Simons theory, obtained via formal, perturbative methods [Hik01, Hik07, Dim13, DGLZ09, DFM11]. Said function is expected to be annihilated by some version of the $\hat{A}$-polynomial; for instance, in [Dim13] this is argued to be the case for the trefoil and figure-eight knot in S^3.

3 Basic definitions and setting

The spaces and their symplectic structures. Throughout this paper, we will write $\mathbb{T}^2$ to denote the real torus $S^1 \times S^1$, and $\mathbb{T}^2_{\mathbb{C}}$ for the 2-dimensional complex torus $\mathbb{C}^* \times \mathbb{C}^*$ containing it. We shall use coordinates $u, v \in \mathbb{R}$ on $\mathbb{T}^2$ and $U, V \in \mathbb{C}$ on $\mathbb{T}^2_{\mathbb{C}}$, with

$$(u,v) \mapsto \left(e^{2\pi i u}, e^{2\pi i v}\right) \in \mathbb{T}^2 , \qquad (U,V) \mapsto \left(e^{2\pi i U}, e^{2\pi i V}\right) \in \mathbb{T}^2_{\mathbb{C}} .$$

We refer to these as the logarithmic coordinates, as opposed to the exponential coordinates on $\mathbb{T}^2_{\mathbb{C}}$

$$m = e^{2\pi i U} , \qquad \ell = e^{2\pi i V} .$$

Moreover, u and v extend to (multivalued) functions on $\mathbb{T}^2_{\mathbb{C}}$ as the real parts of U and V respectively, and together with the imaginary parts they form a coordinate system. This enables us to make sense of $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ as vector fields on $\mathbb{T}^2_{\mathbb{C}}$.

On each space we shall consider a symplectic 2-form:

$$\omega = -2\pi \, \mathrm{d}u \wedge \mathrm{d}v ; \qquad \omega_{\mathbb{C}} = -2\pi \, \mathrm{d}U \wedge \mathrm{d}V .$$

In spite of being closed and non-degenerate, $\omega_{\mathbb{C}}$ does not strictly speaking define a symplectic structure since it is $\mathbb{C}$-valued. However, its real part $\mathrm{Re}(\omega_{\mathbb{C}})$ is still closed and non-degenerate as a real 2-form, thus defining a genuine symplectic structure on $\mathbb{T}^2_{\mathbb{C}}$. Moreover, the restriction of $\omega_{\mathbb{C}}$ to $\mathbb{T}^2$ is ω.

For N a positive integer and S any real number, we call $t = N + iS$ the level of the theory, and define the level t (real) symplectic structure on $\mathbb{T}^2_{\mathbb{C}}$ as

$$\omega_t = \frac{1}{2} \mathrm{Re}\left(t\omega_{\mathbb{C}}\right) .$$

We stress that this restricts to the symplectic structure $N\omega$ on $\mathbb{T}^2$.

The pre-quantum line bundle. For every fixed level t we consider the complex line bundle $\mathscr{L}^{(t)}$ defined by the quasi-periodicity conditions

$$\psi(U+1, V) = e^{-\pi i \,\mathrm{Re}(tV)} \psi(U,V) ,$$
$$\psi(U, V+1) = e^{\pi i \,\mathrm{Re}(tU)} \psi(U,V) .$$

Since these transformations are unitary, the usual Hermitian structure on $\mathbb{C}$ descends to a Hermitian structure $\langle\cdot|\cdot\rangle$ on the bundle. Moreover, we consider on $\mathscr{L}^{(t)}$ the connection $\nabla^{(t)} = \mathrm{d} - i\theta^{(t)}$, where

$$\theta^{(t)}_{(U,V)} = \pi \,\mathrm{Re}\left(t\big(V\,\mathrm{d}U - U\,\mathrm{d}V\big)\right) .$$

It is immediate to check that this connection is compatible with the quasi-periodicity conditions and with the Hermitian structure. Moreover, $\mathrm{d}\theta^{(t)} = \omega_t$, so the curvature of $\nabla^{(t)}$ is $-i\omega_t$, making the triple $(\mathscr{L}^{(t)}, \langle\cdot|\cdot\rangle, \nabla^{(t)})$ into a pre-quantum line bundle for $(\mathbb{T}^2_{\mathbb{C}}, \omega_t)$. The parameter s disappears in the restriction to $\mathbb{T}^2$, which defines a pre-quantum line bundle $(\mathscr{L}^{(N)}, \langle\cdot|\cdot\rangle, \nabla^{(N)})$ for $(\mathbb{T}^2, N\omega)$.

The quasi-periodicity conditions and the connection form for $\nabla^{(N)}$ on $\mathscr{L}^{(N)}$ read explicitly as

$$\begin{aligned}
\psi(u+1,v) &= e^{-N\pi i v}\psi(u,v)\,,\\
\psi(u,v+1) &= e^{N\pi i u}\psi(u,v)\,,\\
\theta^{(N)}_{(u,v)} &= N\pi\big(v\,\mathrm{d}u - u\,\mathrm{d}v\big)\,.
\end{aligned}$$

We refer to these as the Chern-Simons line bundles over $\mathbb{T}^2_{\mathbb{C}}$ and $\mathbb{T}^2$ at the level t respectively, and we shall often omit the superscript in the connection.

The family of complex structures and polarisations. Denote by $\mathcal{T}$ the complex upper half-plane

$$\mathcal{T} = \left\{\ \sigma \in \mathbb{C} : \mathrm{Im}(\sigma) > 0\ \right\}\ .$$

To every point of $\mathcal{T}$ one can associate an almost complex structure on $\mathbb{T}^2$ represented in the logarithmic coordinates by the constant matrix

$$J := \frac{i}{\sigma - \overline{\sigma}} \begin{pmatrix} -(\sigma+\overline{\sigma}) & 2\sigma\overline{\sigma} \\ -2 & \sigma+\overline{\sigma} \end{pmatrix}\ . \tag{12}$$

Similarly, J can be extended to an almost complex structure on $\mathbb{T}^2_{\mathbb{C}}$ by $\mathbb{C}$-*anti*-linearity. We shall also consider its $\mathbb{C}$-linear extension $\overline{J}$. The reason for this seemingly misleading notation will be apparent momentarily. It is straightforward to check that $J^2 = -\mathbb{1}$. Moreover, consider the tensor $g := \omega\cdot J$, which is represented by the matrix

$$g = \frac{2\pi i}{\sigma-\overline{\sigma}}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} -(\sigma+\overline{\sigma}) & 2\sigma\overline{\sigma} \\ -2 & \sigma+\overline{\sigma} \end{pmatrix} = \frac{2\pi i}{\sigma-\overline{\sigma}}\begin{pmatrix} 2 & -(\sigma+\overline{\sigma}) \\ -(\sigma+\overline{\sigma}) & 2\sigma\overline{\sigma} \end{pmatrix}\ .$$

This matrix is symmetric, and it is immediately checked that $\det(g) = 4\pi^2 > 0$, and as the diagonal entries of g are positive the matrix is positive definite. Therefore, g defines a Riemannian metric on $\mathbb{T}^2$, and since it is represented by a matrix with constant entries its Levi-Civita ∇ is the trivial connection in these coordinates, and it is therefore flat. In particular, $\nabla J = 0$, proving that J is integrable and that the triple (ω, g, J) defines a Kähler structure on $\mathbb{T}^2$. Since on $\mathbb{T}^2_{\mathbb{C}}$ the structure J anti-commutes with the multiplication by i, a similar argument also shows that in this case J defines a hyper-Kähler structure.

Consider the eigen-space distributions of $\overline{J}$ on $\mathbb{T}^2_{\mathbb{C}}$, as sub-bundles of the *real* tangent bundle:

$$P = P_\sigma = \left\{\ X \in \mathrm{T}\,\mathbb{T}^2_{\mathbb{C}} : \overline{J}X = -iX\ \right\},\quad \overline{P} = \overline{P}_\sigma = \left\{\ X \in \mathrm{T}\,\mathbb{T}^2_{\mathbb{C}} : \overline{J}X = iX\ \right\}.$$

In spite of sitting inside the *real* tangent bundle $\mathrm{T}\,\mathbb{T}^2_{\mathbb{C}}$, these distributions are defined by the same formal equations as the holomorphic and anti-holomorphic tangent bundles of a Kähler manifold. In particular, by analogous arguments as in the previous situation, P and $\overline{P}$ are polarisations for $\omega_{\mathbb{C}}$, and therefore for ω_t at every level t.

It is convenient to realise these properties of the polarisations explicitly. First of all, define vectors:

$$\begin{aligned} \frac{\partial}{\partial \overline{w}} &:= \frac{\mathbb{1} + iJ}{2}\frac{\partial}{\partial u} = \frac{1}{\sigma - \overline{\sigma}}\left(\sigma\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) \in P\,, \\ \frac{\partial}{\partial w} &:= \frac{\mathbb{1} - iJ}{2}\frac{\partial}{\partial u} = -\frac{1}{\sigma - \overline{\sigma}}\left(\overline{\sigma}\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) \in \overline{P}\,. \end{aligned} \tag{13}$$

Each distribution is spanned over $\mathbb{C}$ by the corresponding section, hence they are Lagrangian (and dual). In logarithmic coordinates, the leaf of P through a point (U, V) is explicitly described as

$$(U, V) + \mathbb{C}\left\langle \frac{1}{\sigma - \overline{\sigma}}(\sigma, 1)\right\rangle\,.$$

Because $\mathrm{Im}(\sigma) > 0$, this leaf only intersects $(U, V) + \mathbb{R}^2 \subseteq \mathbb{C}^2$ at (U, V). Therefore, the exponential map $\mathbb{C}^2 \to \mathbb{T}^2_{\mathbb{C}}$ is injective on each of these sets, so it defines an explicit homeomorphism of each leaf of P with $\mathbb{C}$.

Notice that the same formal relations as in (13), but thought of as in the complexified tangent bundle, define a holomorphic and an anti-holomorphic vector field on $\mathbb{T}^2$. It is convenient to write down the metric and symplectic form on this space in the complex coordinates, for later reference. Because they are both tensors of type $(1, 1)$, they are completely determined by their contraction with $\frac{\partial}{\partial w}$ and $\frac{\partial}{\partial \overline{w}}$. First we have

$$\begin{aligned} \omega\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}}\right) &= \frac{1}{4}\omega\left(\frac{\partial}{\partial u} - iJ\frac{\partial}{\partial u}, \frac{\partial}{\partial u} + iJ\frac{\partial}{\partial u}\right) = \\ &= \frac{i}{2}\omega\left(\frac{\partial}{\partial u}, J\frac{\partial}{\partial u}\right) = -\frac{2\pi}{\sigma - \overline{\sigma}}\,, \end{aligned} \tag{14}$$

which implies

$$\left[\nabla_w, \nabla_{\overline{w}}\right] = \frac{2N\pi i}{\sigma - \overline{\sigma}}\,.$$

Moreover,

$$g\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}}\right) = -i\omega\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}}\right) = \frac{2\pi i}{\sigma - \overline{\sigma}}\,,$$

so one may write the inverse of the metric as

$$\tilde{g} = -i\frac{\sigma - \overline{\sigma}}{2\pi}\left(\frac{\partial}{\partial w} \otimes \frac{\partial}{\partial \overline{w}} + \frac{\partial}{\partial \overline{w}} \otimes \frac{\partial}{\partial w}\right)\,.$$

Therefore, the Laplace operator can be expressed in terms of w and $\overline{w}$ as

$$\Delta = -i\frac{\sigma - \overline{\sigma}}{2\pi}\Big(\nabla_w\nabla_{\overline{w}} + \nabla_{\overline{w}}\nabla_w\Big)\,.$$

Variation tensors and the Hitchin-Witten connection. We shall now compute the variation tensors relevant for the Hitchin-Witten connection. The variation of J is easily calculated by direct computation

$$\begin{aligned} \frac{\partial}{\partial \sigma}J &= -\frac{i}{(\sigma - \overline{\sigma})^2}\begin{pmatrix} -(\sigma + \overline{\sigma}) & 2\sigma\overline{\sigma} \\ -2 & \sigma + \overline{\sigma} = \end{pmatrix} + \frac{i}{\sigma - \overline{\sigma}}\begin{pmatrix} -1 & 2\overline{\sigma} \\ & 1 \end{pmatrix} \\ &= \frac{2i}{(\sigma - \overline{\sigma})^2}\begin{pmatrix} \overline{\sigma} & -\overline{\sigma}^2 \\ 1 & -\overline{\sigma} \end{pmatrix}\,. \end{aligned}$$

In a completely analogous way one finds that

$$\frac{\partial}{\partial\overline{\sigma}}J = \frac{2i}{(\sigma-\overline{\sigma})^2}\begin{pmatrix}-\sigma & \sigma^2\\ -1 & \sigma\end{pmatrix}.$$

The tensor $\tilde{G}$ can now be calculated as contraction of the variation of J with $\tilde{\omega}$

$$\tilde{G}\left(\frac{\partial}{\partial\sigma}\right) = \left(\frac{\partial}{\partial\sigma}J\right)\cdot\tilde{\omega} = \frac{2i}{(\sigma-\overline{\sigma})^2}\begin{pmatrix}\overline{\sigma} & -\overline{\sigma}^2\\ 1 & -\overline{\sigma}\end{pmatrix}\frac{1}{2\pi}\begin{pmatrix} & 1\\ -1 & \end{pmatrix} =$$

$$= \frac{i}{\pi(\sigma-\overline{\sigma})^2}\begin{pmatrix}\overline{\sigma}^2 & \overline{\sigma}\\ \overline{\sigma} & 1\end{pmatrix}$$

and

$$\tilde{G}\left(\frac{\partial}{\partial\overline{\sigma}}\right) = -\frac{i}{\pi(\sigma-\overline{\sigma})^2}\begin{pmatrix}\sigma^2 & \sigma\\ \sigma & 1\end{pmatrix}.$$

The coefficients in these matrices are easily compared to those in ∂_w and $\partial_{\overline{w}}$, so one obtains

$$\tilde{G} = \frac{i}{\pi}\left(\frac{\partial}{\partial w}\otimes\frac{\partial}{\partial w}\otimes d\sigma - \frac{\partial}{\partial\overline{w}}\otimes\frac{\partial}{\partial\overline{w}}\otimes d\overline{\sigma}\right).$$

The splitting of $\tilde{G}$ into its various parts is very transparent in this expression, and shows that J is holomorphic and rigid. In fact, $\tilde{G}$ has constant coefficients, and since the Levi-Civita connection is trivial this implies that $\nabla\tilde{G}(V) = 0$ for very $V \in \mathrm{T}_{\mathbb{C}}\mathcal{T}$. As a consequence, denoting by ∇_w and $\nabla_{\overline{w}}$ the covariant derivative along $\frac{\partial}{\partial w}$ and $\frac{\partial}{\partial\overline{w}}$, one has

$$\Delta_{G(\partial/\partial\sigma)} = \frac{i}{\pi}\nabla_w\nabla_w\,, \qquad \Delta_{G(\partial/\partial\overline{\sigma})} = -\frac{i}{\pi}\nabla_{\overline{w}}\nabla_{\overline{w}}.$$

As was already mentioned, each of these Kähler metrics is flat, which means in particular that the Ricci potential F vanishes. Putting everything together, we obtain the desired expression for the Hitchin-Witten connection

$$\tilde{\boldsymbol{\nabla}}_V = \nabla^{\mathrm{Tr}}_V + \frac{iV'}{2\pi t}\nabla_w\nabla_w + \frac{iV''}{2\pi\bar{t}}\nabla_{\overline{w}}\nabla_{\overline{w}}\,. \tag{15}$$

3.1 Motivational example: the moduli spaces of flat connections on a genus 1 surface

The surface and its moduli spaces. The setting described above includes a model for the moduli spaces $\mathcal{M}$ and $\mathcal{M}_{\mathbb{C}}$ of flat G-connections on a surface of genus 1, for $G = \mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{C})$ respectively. Indeed, call Σ the quotient of $\mathbb{R}^2$ by the lattice $2\pi i\cdot(\mathbb{Z}\times\mathbb{Z})$ with the *reversed* orientation, regarded as a smooth surface, and use 1-periodic coordinates x and y. By the Riemann-Hilbert correspondence, a gauge class of flat G-connections on Σ is completely determined by the class of its holonomy in $\mathrm{Hom}(\pi_1(\Sigma), G)/G$. Any G-representation of $\pi_1(\Sigma)$ takes values in a maximal torus, and is therefore conjugated to one in diagonal matrices. The full quotient is obtained by further moding out by the Weyl group, which acts on each matrix by exchanging its diagonal entries, or equivalently by inverting them. Such matrices are identified by their first entry, so that gauge classes are in bijection with pairs of unitary or invertible complex numbers as the case may be, modulo the relation $(z_1, z_2) \sim (z_1^{-1}, z_2^{-1})$. This finally gives an identification

$$\mathcal{M} \simeq \mathbb{T}^2/\sim , \qquad \mathcal{M}_{\mathbb{C}} \simeq \mathbb{T}^2_{\mathbb{C}}/\sim .$$

As is immediately checked, a pair $(e^{2\pi i\alpha}, e^{2\pi i\beta})$ is explicitly represented by the connection form

$$\alpha T\,\mathrm{d}x + \beta T\,\mathrm{d}y\,,$$

where

$$T = \begin{pmatrix} 2\pi i & 0 \\ 0 & -2\pi i \end{pmatrix}.$$

The Atiyah-Bott form. Each moduli space comes equipped with the Atiyah-Bott symplectic form ω^{AB} or $\omega^{\mathrm{AB}}_{\mathbb{C}}$ as the case may be, which is defined as follows. By linearisation of the relations defining the moduli spaces, their tangent space at a point $[A]$ has a description as a certain cohomology group, whose classes are represented by $\mathfrak{lie}(G)$-valued 1-forms. The wedge product of two such forms gives a 2-form with values in $\mathfrak{lie}(G)^{\otimes 2}$, and an ordinary 2-form can be obtained by means of the Killing pairing. Integration of this does not depend on the choices made, and defines the value of the Atiyah-Bott form up to a normalisation

$$\omega^{\mathrm{AB}}_{(\mathbb{C})}([\eta_1],[\eta_2]) = 4\pi \int_\Sigma \langle \eta_1 \wedge \eta_2 \rangle = -\frac{1}{2\pi}\int_\Sigma \mathrm{tr}(\eta_1 \wedge \eta_2)\,.$$

In the identification above, one has correspondences

$$\frac{\partial}{\partial u} \rightsquigarrow T\,\mathrm{d}x \qquad \text{and} \qquad \frac{\partial}{\partial v} \rightsquigarrow T\,\mathrm{d}y\,.$$

The form is completely determined by the value on these two vectors fields

$$\omega^{\mathrm{AB}}_{(\mathbb{C})}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = \frac{1}{2\pi}\int_\Sigma (\mathrm{tr}\,T^2)\,\mathrm{d}x \wedge \mathrm{d}y = -4\pi\,.$$

Lifting the Atyah-Bott forms to $\mathbb{T}^2$ and $\mathbb{T}^2_{\mathbb{C}}$, it follows from this that

$$\omega^{\mathrm{AB}} = -4\pi du \wedge dv = 2\omega\,, \qquad \omega^{\mathrm{AB}}_{\mathbb{C}} = -4\pi dU \wedge dV = 2\omega_{\mathbb{C}}\,.$$

The Chern-Simons line bundle. Chern-Simons theory defines a level $t = k + is$ pre-quantum line bundle $\mathscr{L}^{\mathrm{CS},(k+is)}$ on $\mathcal{M}_{\mathbb{C}}$ for every $k \in \mathbb{Z}_{>0}$ and $s \in \mathbb{R}$. The construction is similar to the one for the case of compact groups, described in great detail in [Fre95]. The bundle $\mathscr{L}^{(k+is)}$ is defined by considering first the trivial Hermitian line bundle $\mathcal{A}_{\mathrm{SL}(2,\mathbb{C})} \times \mathbb{C}$ on the space $\mathcal{A}_{\mathrm{SL}(2,\mathbb{C})}$ of all $\mathrm{SL}(2,\mathbb{C})$-connections. The Chern-Simons action induces a lift of the action of the gauge group to the bundle and a connection $\nabla^{\mathrm{CS},(k+is)}$ expressed by the form

$$\theta^{\mathrm{CS},(k+is)}_A(\eta) = -2\pi \int_\Sigma \mathrm{Re}\Big((k+is)\,\langle A \wedge \eta\rangle\Big) = \frac{1}{4\pi}\int_\Sigma \mathrm{Re}\Big((k+is)\,\mathrm{tr}(A\wedge\eta)\Big)\,.$$

One may then restrict to the subspace $\mathcal{F}_{\mathrm{SL}(2,\mathbb{C})} \subseteq \mathcal{A}_{\mathrm{SL}(2,\mathbb{C})}$ consisting of flat connections and mod out by the gauge group.

For a description of this structure in coordinates, one can think of the universal cover of $\mathbb{T}^2_{\mathbb{C}}$ as sitting inside $\mathcal{F}_G$ for the relevant G. As it turns out, the stabiliser of $\mathbb{T}^2_{\mathbb{C}}$ in the gauge group is generated by three elements, which act on the bundle as

$$\begin{aligned} g_U\colon \big((U,V),\psi\big) &\mapsto \Big((U+1,V), e^{-2\pi i\,\mathrm{Re}\big((k+is)V\big)}\psi\Big)\,,\\ g_V\colon \big((U,V),\psi\big) &\mapsto \Big((U,V+1), e^{2\pi i\,\mathrm{Re}\big((k+is)U\big)}\psi\Big)\,,\\ g_-\colon \big((U,V),\psi\big) &\mapsto \big((-U,-V),\psi\big)\,. \end{aligned}$$

Moreover, a quick computation shows that the connection form reads explicitly

$$\theta^{\mathrm{CS},(k+is)}_{(U,V)} = 2\pi\,\mathrm{Re}\Big((k+is)\big(V\,\mathrm{d}U - U\,\mathrm{d}V\big)\Big) = \theta^{\big(2(k+is)\big)}\,.$$

Of course this bundle restricts to one on $\mathcal{M}$, which sits naturally inside $\mathcal{M}_{\mathbb{C}}$.

It is apparent from this description that the bundle induced by the Chern-Simons theory on $\mathcal{M}$ and $\mathcal{M}_{\mathbb{C}}$ lifts to the one described above on $\mathbb{T}^2$ and $\mathbb{T}^2_{\mathbb{C}}$ for $t = 2(k+is)$.

Teichmüller space, complex structures and polarisations. For a complex number σ in the upper half plane $\mathbb{H} \subseteq \mathbb{C}$ consider the map $z_\sigma : \mathbb{R}^2 \to \mathbb{C}$ sending (x,y) to $x + \sigma^{-1}y$, which is orientation preserving in our convention. The natural Kähler structure on $\mathbb{C}$ pulls back to one on $\mathbb{R}^2$ which is compatible with the action of $\mathbb{Z}^2$, thus defining one on Σ with $z_\sigma = x+\sigma^{-1}y$ as holomorphic coordinate. This produces a bijection between $\mathbb{H}$ and the Teichmüller space $\mathcal{T}$, whose complex structure makes this into a biholomorphism.

The volume coming from this structure equals the area of the parallelogram spanned by 1 and σ^{-1} in $\mathbb{C}$, which equals $\mathrm{Im}\,\sigma/|\sigma|^2$. It is convenient to rescale the metric by this factor, so as to obtain a torus with volume 1. The resulting metric μ, together with its inverse $\tilde{\mu}$, can be explicitly expressed by the matrices

$$\mu = \frac{i}{\sigma-\overline{\sigma}}\begin{pmatrix} 2\sigma\overline{\sigma} & \sigma+\overline{\sigma} \\ \sigma+\overline{\sigma} & 2 \end{pmatrix}\,, \qquad \tilde{\mu} = \frac{i}{\sigma-\overline{\sigma}}\begin{pmatrix} 2 & -(\sigma+\overline{\sigma}) \\ -(\sigma+\overline{\sigma}) & 2\sigma\overline{\sigma} \end{pmatrix}\,.$$

By construction, the volume form is just $-\,\mathrm{d}x\wedge\mathrm{d}y$.

With μ comes the Hodge $*$ operator, defined by

$$\varphi_1 \wedge *\varphi_2 = -\,\langle\varphi_1|\varphi_2\rangle\,\mathrm{d}x\wedge\mathrm{d}y\,.$$

A straightforward check shows that

$$\mathrm{d}x\wedge *\,\mathrm{d}x = -\frac{2i}{\sigma-\overline{\sigma}}\,\mathrm{d}x\wedge\mathrm{d}y\,, \qquad \mathrm{d}y\wedge *\,\mathrm{d}x = \frac{i(\sigma+\overline{\sigma})}{\sigma-\overline{\sigma}}\,\mathrm{d}x\wedge\mathrm{d}y\,,$$

$$\mathrm{d}x\wedge *\,\mathrm{d}y = \frac{i(\sigma+\overline{\sigma})}{\sigma-\overline{\sigma}}\,\mathrm{d}x\wedge\mathrm{d}y\,, \qquad \mathrm{d}y\wedge *\,\mathrm{d}y = -\frac{2i\sigma\overline{\sigma}}{\sigma-\overline{\sigma}}\,\mathrm{d}x\wedge\mathrm{d}y\,.$$

from which

$$\begin{aligned} *\,\mathrm{d}x &= -\frac{\sigma+\overline{\sigma}}{\sigma-\overline{\sigma}}i\,\mathrm{d}x - \frac{2}{\sigma-\overline{\sigma}}i\,\mathrm{d}y\,,\\ *\,\mathrm{d}y &= \frac{2\sigma\overline{\sigma}}{\sigma-\overline{\sigma}}i\,\mathrm{d}x + \frac{\sigma+\overline{\sigma}}{\sigma-\overline{\sigma}}i\,\mathrm{d}y\,. \end{aligned}$$

Hodge theory can be used to show that the cohomology group representing $\mathrm{T}_{[A]}\,\mathcal{M}$ is isomorphic to the space of harmonic $\mathfrak{su}(n)$-valued forms. This space is preserved by the Hodge star operator, which moreover satisfies $*^2 = -\mathbb{1}$. This defines an almost complex structure J_σ (or simply J) by $J\eta = -*\overline{\eta}$, which is just $*\eta$ on the representatives we are considering. In coordinates (u, v), the corresponding matrix is

$$J = \frac{i}{\sigma - \overline{\sigma}} \begin{pmatrix} -(\sigma + \overline{\sigma}) & 2\sigma\overline{\sigma} \\ -2 & \sigma + \overline{\sigma} \end{pmatrix} .$$

We stress that the lifts of these families of complex structures to $\mathbb{T}^2$ and $\mathbb{T}^2_{\mathbb{C}}$ agree with those already considered in (12). In particular, the same conclusions hold about integrability and flatness, and hence one finds the same variation tensors.

Under the identification of vectors on $\mathcal{M}$ with harmonic $\mathfrak{su}(2)$-valued forms, the complexified tangent spaces correspond to forms with values in $\mathfrak{su}(2) \otimes \mathbb{C} \simeq \mathfrak{sl}(2, \mathbb{C})$. It can be shown, using the properties of the Hodge star-operator, that vectors of type $(1, 0)$ correspond to $(0, 1)$-forms and vice-versa. On the other hand, harmonic $\mathfrak{sl}(2, \mathbb{C})$-valued forms correspond to real vectors on $\mathcal{M}_{\mathbb{C}}$. In this language, Witten uses the polarisation P consisting of holomorphic forms on Σ; again, this agrees with the choice made above for $\mathbb{T}^2_{\mathbb{C}}$.

This concludes our argument that the structure described above encodes the setting of $\mathrm{SL}(2, \mathbb{C})$ Chern-Simons theory for genus 1. In particular, the Hitchin-Witten connection obtained here also agrees with the one considered in [Wit91]. However, this picture is more restrictive than that on $\mathbb{T}^2$ and $\mathbb{T}^2_{\mathbb{C}}$, as it requires that N is even.

4 Operators from geometric quantisation on $\mathbb{T}^2_{\mathbb{C}}$

4.1 The quantum operators on $\mathcal{H}^{(t)}_\sigma$

We shall now study the level-t pre-quantum operators associated to the logarithmic coordinates on $\mathbb{T}^2_{\mathbb{C}}$. Notice that, since U and V are multi-valued functions, these operators are not well defined in principle, unless a fixed branch is specified.

We now fix the level t and $\sigma \in \mathcal{T}$, and introduce coordinates adapted to the polarisation. We consider on the leaf through each point $(u, v) \in \mathbb{T}^2$ the complex coordinate $\xi + i\eta$ induced by its description as $(u, v) + \mathbb{C}\,\langle \frac{\partial}{\partial \overline{w}} \rangle$. The coordinates on $\mathbb{T}^2$, restricted to $0 \leq u, v < 1$, can be uniquely extended to real functions u_σ and v_σ on $\mathbb{T}^2_{\mathbb{C}}$ which are constant along P_σ. Although these functions are non-continuous, their differential is well defined on the whole $\mathbb{T}^2_{\mathbb{C}}$, and hence so is their Hamiltonian vector field. Together with ξ and η these form global real coordinates on the complex torus, and it follows from the definitions that one may write U and V as

$$U = u_\sigma + \frac{\sigma}{\sigma - \overline{\sigma}} \big(\xi + i\eta\big) , \qquad V = v_\sigma + \frac{1}{\sigma - \overline{\sigma}} \big(\xi + i\eta\big) .$$

Notice that u_σ enters (U, V) only through $u = \mathrm{Re}(U)$, so

$$\frac{\partial}{\partial u_\sigma} \mathrm{Im}(U) = \frac{\partial}{\partial u_\sigma} \mathrm{Re}(V) = \frac{\partial}{\partial u_\sigma} \mathrm{Im}(V) = 0 .$$

From this, and the corresponding argument for v_σ, one can conclude that

$$\frac{\partial}{\partial u_\sigma} = \frac{\partial}{\partial u}\,, \qquad \frac{\partial}{\partial v_\sigma} = \frac{\partial}{\partial v}\,.$$

In these coordinates, the complex symplectic form may be written as

$$\omega_{\mathbb{C}} = -2\pi\,\mathrm{d}u_\sigma \wedge \mathrm{d}v_\sigma - \frac{2\pi}{\sigma - \overline{\sigma}}\,\mathrm{d}\big(u_\sigma + \sigma v_\sigma\big) \wedge \mathrm{d}\big(\xi + i\eta\big)\,.$$

It is clear from this that all the Poisson brackets of the coordinates with respect to ω_t are constant on $\mathbb{T}^2_{\mathbb{C}}$. Since every $f \in \mathcal{C}^\infty(\mathbb{T}^2_{\mathbb{C}})$ constant on the leaves is a function of u_σ and v_σ, this is enough to conclude that the bracket of any such function with ξ or η is of the same kind. In conclusion, all four coordinates are affine linear on the leaves, so their pre-quantum operators preserve the space of polarised sections of $\mathscr{L}^{(t)}$, and can be promoted to quantum operators.

Theorem 7. *The quantum operators $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$ act on the smooth sections of $\mathscr{L}^{(N)}$ over $\mathbb{T}^2$ as*

$$\widehat{U}_\sigma = u - \frac{i\sigma}{\pi t}\nabla_w\,, \qquad \widehat{V}_\sigma = v - \frac{i}{\pi t}\nabla_w\,.$$

Proof. By linearity, $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$ are determined once the quantum operators associated to the coordinates u_σ, v_σ, ξ and η are known.

Recall that the pre-quantum operator of a smooth, real function $f \in \mathcal{C}^\infty(\mathbb{T}^2_{\mathbb{C}})$ is defined on sections of $\mathscr{L}^{(t)}$ as

$$\widehat{f} := f - i\nabla_{H_f^{(t)}}\,,$$

where $H_f^{(t)}$, the Hamiltonian vector field of f relative to ω_t, is characterised by

$$X[f] = \omega_t\big(X, H_f^{(t)}\big) \quad \text{for every } X \in \mathrm{T}\,\mathbb{T}^2_{\mathbb{C}}.$$

If a vector field $\tilde{H}$ satisfies this condition for every X tangent to P, then $\tilde{H} - H_f^{(t)}$ is orthogonal, hence tangent, to P. Therefore, the covariant derivatives along $\tilde{H}$ and $H_f^{(t)}$ act in the same way on polarised sections, so the quantum operator $\widehat{f_\sigma}$ may as well be defined using $\tilde{H}$ in place of $H_f^{(t)}$. We shall look for such a $\tilde{H}$ as a linear combination of $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ for each of the coordinates.

Since u_σ and v_σ are constant along P, their Hamiltonian vector fields are tangent to the polarisation. As a consequence, the first-order part of their quantum operators vanish, leading to

$$\widehat{u_\sigma} = u_\sigma\,, \qquad \widehat{v_\sigma} = v_\sigma\,.$$

For real functions α and β, and $\xi, \eta \in \mathbb{R}$ fixed, one can compute

$$\omega_t\left((\xi + i\eta)\frac{\partial}{\partial \overline{w}}, \alpha\frac{\partial}{\partial u} + \beta\frac{\partial}{\partial v}\right) = 2\pi\,\mathrm{Re}\left(t(\xi + i\eta)\frac{\alpha - \sigma\beta}{\sigma - \overline{\sigma}}\right) = \\ = 2\pi\left(\mathrm{Re}\Big(t\frac{\alpha - \sigma\beta}{\sigma - \overline{\sigma}}\Big)\xi - \mathrm{Im}\Big(t\frac{\alpha - \sigma\beta}{\sigma - \overline{\sigma}}\Big)\eta\right).$$

In order to determine $\widehat{\xi}_\sigma$, we look for real functions α and β so that the above equals ξ for every ξ and η. This is equivalent to

$$2\pi t \frac{\alpha - \sigma\beta}{\sigma - \overline{\sigma}} = 1\,.$$

A simple algebraic manipulation leads to

$$\alpha = -\frac{1}{2\pi}\left(\frac{\overline{\sigma}}{t} + \frac{\sigma}{\overline{t}}\right)\,, \qquad \beta = -\frac{1}{2\pi}\left(\frac{1}{t} + \frac{1}{\overline{t}}\right)\,.$$

Therefore we have that

$$\widehat{\xi}_\sigma = \xi + \frac{i}{2\pi t}\big(\overline{\sigma}\nabla_u + \nabla_v\big) + \frac{i}{2\pi \overline{t}}\big(\sigma\nabla_u + \nabla_v\big) = \xi - \frac{i(\sigma - \overline{\sigma})}{2\pi t}\nabla_w + \frac{i(\sigma - \overline{\sigma})}{2\pi \overline{t}}\nabla_{\overline{w}}\,.$$

The operator $\widehat{\eta}_\sigma$ is obtained in the same way: we look for α and β so that

$$2\pi t \frac{\alpha\sigma\beta}{\sigma - \overline{\sigma}} = -i\,,$$

which gives

$$\alpha = \frac{i}{2\pi}\left(\frac{\overline{\sigma}}{t} - \frac{\sigma}{\overline{t}}\right) \qquad \beta = \frac{i}{2\pi}\left(\frac{1}{t} - \frac{1}{\overline{t}}\right)\,.$$

From this we find that

$$\widehat{\eta}_\sigma = \eta + \frac{1}{2\pi t}\big(\overline{\sigma}\nabla_u + \nabla_v\big) - \frac{1}{2\pi \overline{t}}\big(\sigma\nabla_u + \nabla_v\big) = \eta - \frac{\sigma - \overline{\sigma}}{2\pi t}\nabla_w - \frac{\sigma - \overline{\sigma}}{2\pi \overline{t}}\nabla_{\overline{w}}\,.$$

Putting everything together, the quantum operators $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$ act on polarised sections as

$$\widehat{U}_\sigma = U - \frac{i\sigma}{\pi t}\nabla_w\,, \qquad \widehat{V}_\sigma = V - \frac{i}{\pi t}\nabla_w\,.$$

The action on sections on $\mathbb{T}^2$ is obtained by restricting U and V to u and v. Since ∇_w is a linear combination of ∇_u and ∇_v, and since $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ are tangent to $\mathbb{T}^2$, this operator is unchanged under the restriction. This concludes the proof. ☐

Having established this, it is no longer necessary to consider $\mathbb{T}^2_{\mathbb{C}}$, and we may focus our attention to the picture on $\mathbb{T}^2$ instead.

We now wish to define quantum operators for the exponential coordinates m and ℓ, to which end we rely on the spectral theorem for normal operators, see e.g. [Con94]. A densely defined operator N on a separable Hilbert space is called normal if it is closed and $NN^\dagger = N^\dagger N$, where $N^\dagger$ denotes the adjoint of N and the identity includes the equality of the domains. For instance, if X is a measured space of σ-finite measure, f a complex-valued measurable function on X, then the multiplication by f defines a normal operator on $\mathrm{L}^2(X, \mathbb{C})$. As a matter of fact, the theorem states that every normal operator is unitarily equivalent to one of this kind. One remarkable consequence of this is that the exponential series of a normal operator is strongly convergent on a suitable dense subspace.

Lemma 8. *The quantum operators $\widehat{U}_\sigma, \widehat{V}_\sigma \in \mathcal{H}^{(t)}_\sigma$ are normal.*

Proof. It follows form general, standard arguments that covariant derivatives are operators on $\mathrm{L}^2(\mathbb{T}^2, \mathscr{L}^{(N)})$ defined on the same dense domains as their adjoints. On the other hand, in our convention the functions u and v range through $[0, 1)$, so their multiplication operators on $\mathrm{L}^2(\mathbb{T}^2, \mathscr{L}^{(N)})$ are everywhere defined and bounded. From this it follows hat $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$ are also closed, and have the same dense domain as ∇_w. Moreover, it is immediate to check that $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$ also have the same image as their adjoints, so

$$\mathrm{dom}\left(\widehat{U}_\sigma \widehat{U}_\sigma^\dagger\right) = \mathrm{dom}\left(\widehat{U}_\sigma^\dagger \widehat{U}_\sigma\right), \qquad \mathrm{dom}\left(\widehat{V}_\sigma \widehat{V}_\sigma^\dagger\right) = \mathrm{dom}\left(\widehat{V}_\sigma^\dagger \widehat{V}_\sigma\right).$$

We now proceed to checking the commutation relation for $\widehat{U}_\sigma$ and $\widehat{U}_\sigma^\dagger$ by direct computation

$$\begin{aligned}\left[\widehat{U}_\sigma, \widehat{U}_\sigma^\dagger\right] &= \left[u - \frac{i\sigma}{\pi t}\nabla_w, u - \frac{i\overline{\sigma}}{\pi \overline{t}}\nabla_{\overline{w}}\right] = \\ &= \frac{i\sigma}{\pi t} \cdot \frac{\overline{\sigma}}{\sigma - \overline{\sigma}} + \frac{i\overline{\sigma}}{\pi \overline{t}} \cdot \frac{\sigma}{\sigma - \overline{\sigma}} - \frac{\sigma\overline{\sigma}}{\pi^2 t\overline{t}} \cdot \frac{2N\pi i}{\sigma - \overline{\sigma}} = 0\,.\end{aligned}$$

Analogously we find that

$$\begin{aligned}\left[\widehat{V}_\sigma, \widehat{V}_\sigma^\dagger\right] &= \left[v - \frac{i}{\pi t}\nabla_w, v - \frac{i}{\pi \overline{t}}\nabla_{\overline{w}}\right] = \\ &= \frac{i}{\pi t} \cdot \frac{\overline{\sigma}}{\sigma - \overline{\sigma}} + \frac{i}{\pi \overline{t}} \cdot \frac{\sigma}{\sigma - \overline{\sigma}} - \frac{1}{\pi^2 t\overline{t}} \cdot \frac{2N\pi i}{\sigma - \overline{\sigma}} = 0\,.\end{aligned}$$

□

It is easily seen that for every $\lambda \in \mathbb{C}$ and every normal operator N on a Hilbert space λN is also normal. The lemma ensures then that the following is well posed.

Definition 4. *We define quantum operators associated to m and ℓ on $\mathcal{H}_\sigma^{(t)}$ as*

$$\widehat{m}_\sigma = \exp\left(2\pi i \widehat{U}_\sigma\right), \qquad \widehat{\ell}_\sigma = \exp\left(2\pi i \widehat{V}_\sigma\right).$$

The next result shows that the quantum operators found above are compatible with the identification of the various Hilbert space via parallel transport.

Proof of Theorem 1. We need to show that $\widehat{U}_\sigma$ and $\widehat{V}_\sigma$ are covariantly constant with respect to the connection $\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}$ induced on $\mathrm{End}(\mathcal{H}^{(t)})$ by $\tilde{\boldsymbol{\nabla}}$. The covariant derivative of an endomorphisms E of $\mathcal{H}$ along a direction V on $\mathcal{T}$ is given by its commutator with $\tilde{\boldsymbol{\nabla}}_V$

$$\tilde{\boldsymbol{\nabla}}_V^{\mathrm{End}}(E) = \left[\tilde{\boldsymbol{\nabla}}_V, E\right].$$

It is a consequence of (14) that the commutator of the covariant derivatives along ∂_w and $\partial_{\overline{w}}$ is

$$\left[\nabla_w, \nabla_{\overline{w}}\right] = -iN\omega\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}}\right) = \frac{2N\pi i}{\sigma - \overline{\sigma}}.$$

We start with the Hitchin-Witten derivatives of the multiplication by u

$$\begin{aligned}\tilde{\boldsymbol{\nabla}}_\sigma^{\mathrm{End}} u &= \frac{\partial}{\partial \sigma}u + \frac{i}{2\pi t}\left[\nabla_w \nabla_w, u\right] = -\frac{i\overline{\sigma}}{\pi t(\sigma - \overline{\sigma})}\nabla_w\,, \\ \tilde{\boldsymbol{\nabla}}_{\overline{\sigma}}^{\mathrm{End}} u &= \frac{\partial}{\partial \overline{\sigma}}u + \frac{i}{2\pi \overline{t}}\left[\nabla_{\overline{w}} \nabla_{\overline{w}}, u\right] = \frac{i\sigma}{\pi \overline{t}(\sigma - \overline{\sigma})}\nabla_{\overline{w}}\,.\end{aligned}$$

Analogously for v

$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\sigma} v = \frac{\partial}{\partial\sigma} v + \frac{i}{2\pi t}\big[\nabla_w\nabla_w, v\big] = -\frac{i}{\pi t(\sigma-\overline{\sigma})}\nabla_w\,,$$
$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\overline{\sigma}} v = \frac{\partial}{\partial\overline{\sigma}} v + \frac{i}{2\pi \overline{t}}\big[\nabla_{\overline{w}}\nabla_{\overline{w}}, v\big] = \frac{i}{\pi \overline{t}(\sigma-\overline{\sigma})}\nabla_{\overline{w}}\,.$$

Consider next the variation of ∇_w in σ

$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\sigma}(\nabla_w) = \frac{\partial}{\partial\sigma}(\nabla_w) + \frac{i}{2\pi t}\big[\nabla_w\nabla_w, \nabla_w\big] = -\frac{\partial}{\partial\sigma}\left(\frac{\overline{\sigma}\nabla_u+\nabla_v}{\sigma-\overline{\sigma}}\right) =$$
$$= \frac{\overline{\sigma}\nabla_u+\nabla_v}{(\sigma-\overline{\sigma})^2} = -\frac{1}{\sigma-\overline{\sigma}}\nabla_w\,.$$

The variation in $\overline{\sigma}$ is more involved

$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\overline{\sigma}}(\nabla_w) = \frac{\partial}{\partial\overline{\sigma}}(\nabla_w) + \frac{i}{2\pi \overline{t}}\big[\nabla_{\overline{w}}\nabla_{\overline{w}}, \nabla_w\big] =$$
$$= -\frac{\partial}{\partial\overline{\sigma}}\left(\frac{\overline{\sigma}\nabla_u+\nabla_v}{\sigma-\overline{\sigma}}\right) + \frac{i}{\pi\overline{t}}\big[\nabla_{\overline{w}}, \nabla_w\big]\nabla_{\overline{w}} =$$
$$= -\frac{\overline{\sigma}\nabla_u+\nabla_v}{(\sigma-\overline{\sigma})^2} - \frac{1}{\sigma-\overline{\sigma}}\nabla_u + \frac{2N}{\overline{t}(\sigma-\overline{\sigma})}\nabla_{\overline{w}} =$$
$$= -\frac{1}{\overline{t}(\sigma-\overline{\sigma})}\left(\overline{t}\frac{\sigma\nabla_u+\nabla_v}{\sigma-\overline{\sigma}} - 2N\nabla_{\overline{w}}\right) = \frac{t}{\overline{t}(\sigma-\overline{\sigma})}\nabla_{\overline{w}}\,.$$

Altogether, this shows the covariant derivative of $\widehat{U}_\sigma$ with respect to σ is

$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\sigma}(\widehat{U}_\sigma) = \tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\sigma}\left(u - \frac{i\sigma}{\pi t}\nabla_w\right) =$$
$$= \tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\sigma}(u) - \frac{i}{\pi t}\nabla_w - \frac{i\sigma}{\pi t}\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\sigma}(\nabla_w) =$$
$$= -\frac{i}{\pi t}\left(\frac{\overline{\sigma}}{\sigma-\overline{\sigma}} + 1 - \frac{\sigma}{\sigma-\overline{\sigma}}\right)\nabla_w = 0\,.$$

In $\overline{\sigma}$ we have that

$$\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\overline{\sigma}}(\widehat{U}_\sigma) = \tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\overline{\sigma}}(u) - \frac{i\sigma}{\pi t}\tilde{\boldsymbol{\nabla}}^{\mathrm{End}}_{\overline{\sigma}}(\nabla_w) =$$
$$= \frac{i\sigma}{\pi\overline{t}(\sigma-\overline{\sigma})}\nabla_{\overline{w}} - \frac{i\sigma}{\pi t}\frac{t}{\overline{t}(\sigma-\overline{\sigma})}\nabla_{\overline{w}} = 0\,.$$

The derivatives of $\widehat{V}_\sigma$ are completely analogous. □

4.2 Trivialisation of the Hitchin-Witten connection and σ-independent operators

It was established in the previous paragraph that the quantum operators found there are independent on the Teichmüller parameter in the sense of the Hitchin-Witten connection. Our next goal is to remove the dependence on σ explicitly. To this end, we shall use the following trivialisation of the Hitchin-Witten connection, based on Witten's proposal in [Wit91].

Theorem 9. *The Hitchin-Witten connection for $\mathbb{T}^2$ is unitarily gauge-equivalent to the trivial connection, and the equivalence is realised as*

$$\exp\big(r\Delta\big)\,\tilde{\boldsymbol{\nabla}}\,\exp\big(-r\Delta\big) = \nabla^{Tr},$$

where $r \in \mathbb{C}$ is such that

$$e^{4rN} = -\frac{\overline{t}}{t}\,. \tag{16}$$

Proof. Since $\left|\overline{t}/t\right| = 1$, r is purely imaginary. That the gauge transformation $\exp(r\Delta)$ is unitary follows from this and that Δ is a self-adjoint operator.

In order to prove the statement it is enough to show that

$$\exp\left(-r\Delta\right)\frac{\partial}{\partial\sigma}\big[\exp\left(r\Delta\right)\big] = \frac{1}{2t}b\Big(\frac{\partial}{\partial\sigma}\Big)\,,$$

$$\exp\left(-r\Delta\right)\frac{\partial}{\partial\overline{\sigma}}\big[\exp\left(r\Delta\right)\big] = -\frac{1}{2\overline{t}}\overline{b}\Big(\frac{\partial}{\partial\overline{\sigma}}\Big)\,.$$

We now proceed by deriving the exponential series of $r\Delta$ term-wise along $\frac{\partial}{\partial\sigma}$. Recall that the derivative of Δ is given by $-\Delta_{\tilde{G}}$, so

$$\frac{\partial}{\partial\sigma}\Delta = -\Delta_{\tilde{G}\left(\frac{\partial}{\partial\sigma}\right)} = -\frac{i}{\pi}\nabla_w\nabla_w = -b\Big(\frac{\partial}{\partial\sigma}\Big)\,.$$

It is useful to compute the commutator

$$\begin{aligned}\left[\frac{\partial}{\partial\sigma}\Delta,\Delta\right] &= -\frac{\sigma-\overline{\sigma}}{2\pi^2}\Big[\nabla_w\nabla_w,\nabla_w\nabla_{\overline{w}}+\nabla_{\overline{w}}\nabla_w\Big] = \\ &= -2\frac{\sigma-\overline{\sigma}}{\pi^2}\left[\nabla_w,\nabla_{\overline{w}}\right]\nabla_w\nabla_w = -\frac{4Ni}{\pi}\nabla_w\nabla_w = 4N\frac{\partial}{\partial\sigma}\Delta\,,\end{aligned}$$

and it is checked by induction that furthermore

$$\left[\frac{\partial}{\partial\sigma}\Delta,\Delta^n\right] = \sum_{l=1}^{n}\binom{n}{l}(4N)^l\Delta^{n-l}\frac{\partial}{\partial\sigma}\Delta\,.$$

This can be used to compute the derivative of Δ^n:

$$\begin{aligned}\frac{\partial}{\partial\sigma}\big(\Delta^n\big) &= \sum_{j=1}^{n}\Delta^{n-j}\frac{\partial\Delta}{\partial\sigma}\Delta^{j-1} = \\ &= n\Delta^{n-1}\frac{\partial\Delta}{\partial\sigma} + \sum_{j=1}^{n}\sum_{l=1}^{j-1}\binom{j-1}{l}(4N)^l\Delta^{n-l-1}\frac{\partial\Delta}{\partial\sigma}\,.\end{aligned}$$

One can now exchange the sums and use the identity

$$\sum_{j=l+1}^{n}\binom{j-1}{l} = \binom{n}{l+1}\,,$$

and after incorporating the single term on the left one finds

$$\frac{\partial}{\partial\sigma}\big(\Delta^n\big) = \sum_{l=0}^{n-1}\binom{n}{l+1}(4N)^l\Delta^{n-l-1}\frac{\partial\Delta}{\partial\sigma}\,.$$

We now apply this to the derivative of the exponential series

$$\frac{\partial}{\partial\sigma}\exp\bigl(r\Delta\bigr) = \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\binom{n}{l+1}\frac{(4N)^l r^n}{n!}\Delta^{n-l-1}\frac{\partial\Delta}{\partial\sigma}\,.$$

We now change l with $n-l-1$, switch the integrals and further change n with $n+l$ to find

$$\begin{aligned}\frac{\partial}{\partial\sigma}\exp\bigl(r\Delta\bigr) &= \sum_{l=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n!}\frac{1}{l!}(4N)^{n-1}r^{n+l}\Delta^l\frac{\partial\Delta}{\partial\sigma} = \\ &= \left(\frac{1}{2N}\sum_{n=1}^{\infty}\frac{(4Nr)^n}{n!}\right)\sum_{l=0}^{\infty}\frac{(r\Delta)^l}{l!}\frac{\partial\Delta}{\partial\sigma}\,.\end{aligned}$$

The sum on the left is the exponential series at $4Nr$, but starting at $n=1$, so it gives

$$\frac{e^{4Nr}-1}{4N} = -\frac{1}{4N}\left(\frac{\bar{t}}{t}+1\right) = -\frac{1}{2t}$$

Finally, this gives

$$\frac{\partial}{\partial\sigma}\exp\bigl(r\Delta\bigr) = -\frac{1}{2t}\exp\bigl(r\Delta\bigr)\frac{\partial\Delta}{\partial\sigma} = \frac{1}{2t}\exp\bigl(r\Delta\bigr)b\Bigl(\frac{\partial}{\partial\sigma}\Bigr)\,.$$

This concludes the proof for $\frac{\partial}{\partial\sigma}$. The argument for $\frac{\partial}{\partial\bar{\sigma}}$ is analogous. □

Definition 5. *We define the σ-independent quantum operators associated to U, V, m and ℓ to be*

$$\begin{aligned}\widehat{U} &= \exp\bigl(r\Delta\bigr)\widehat{U}_\sigma\bigl(-r\Delta\bigr)\,, & \widehat{V} &= \exp\bigl(r\Delta\bigr)\widehat{V}_\sigma\bigl(-r\Delta\bigr)\,,\\ \widehat{m} &= \exp\bigl(r\Delta\bigr)\widehat{m}_\sigma\bigl(-r\Delta\bigr)\,, & \widehat{\ell} &= \exp\bigl(r\Delta\bigr)\widehat{\ell}_\sigma\bigl(-r\Delta\bigr)\,.\end{aligned}$$

Theorem 10. *The σ-independent operators are expressed by*

$$\widehat{U} = u - i\frac{e^{2rN}-1}{2N\pi}\nabla_v \qquad \widehat{V} = v + i\frac{e^{2rN}-1}{2N\pi}\nabla_u\,,$$

$$\widehat{m} = \exp\bigl(2\pi i\widehat{U}\bigr) = e^{2\pi iu}\exp\left(\frac{e^{2rN}-1}{N}\nabla_v\right),$$

$$\widehat{\ell} = \exp\bigl(2\pi i\widehat{V}\bigr) = e^{2\pi iv}\exp\left(-\frac{e^{2rN}-1}{N}\nabla_u\right).$$

We wish to stress how these operators are manifestly independent of the Teichmüller parameter.

Proof. We shall make use of the following formula:

$$\mathrm{Ad}_{\exp(r\Delta)} = \exp\bigl(r\,\mathrm{ad}_\Delta\bigr)\,.$$

According to this, in order to conjugate the operators by $\exp(r\Delta)$ it is enough to understand their commutator with Δ.

We start by computing the commutator of Δ with u

$$\big[\Delta, u\big] = -i\frac{\sigma-\overline{\sigma}}{\pi}\left(\frac{\sigma}{\sigma-\overline{\sigma}}\nabla_w - \frac{\overline{\sigma}}{\sigma-\overline{\sigma}}\nabla_{\overline{w}}\right) = -\frac{i}{\pi}\big(\sigma\nabla_w - \overline{\sigma}\nabla_{\overline{w}}\big)\,.$$

Similarly we have

$$\big[\Delta, v\big] = -i\frac{\sigma-\overline{\sigma}}{\pi}\left(\frac{1}{\sigma-\overline{\sigma}}\nabla_w - \frac{1}{\sigma-\overline{\sigma}}\nabla_{\overline{w}}\right) = -\frac{i}{\pi}\big(\nabla_w - \nabla_{\overline{w}}\big)\,.$$

The action of ad^n_Δ on u and v is then determined by that on ∇_w and $\nabla_{\overline{w}}$. This is easily determined by the following relations

$$\begin{aligned}\big[\Delta, \nabla_w\big] &= -\frac{\sigma-\overline{\sigma}}{\pi}\frac{2N\pi}{\sigma-\overline{\sigma}}\nabla_w = -2N\nabla_w\,,\\ \big[\Delta, \nabla_{\overline{w}}\big] &= \frac{\sigma-\overline{\sigma}}{\pi}\frac{2N\pi}{\sigma-\overline{\sigma}}\nabla_{\overline{w}} = 2N\nabla_{\overline{w}}\,.\end{aligned}$$

Given this, we can move on to computing the conjugation of the operators. For the multiplication by u we find

$$\begin{aligned}\exp\big(r\Delta\big)u\exp\big(-r\Delta\big) &= \sum_{n=0}^{\infty}\frac{r^n}{n!}\,\mathrm{ad}^n_\Delta(u) =\\ &= u - \frac{i}{\pi}\sum_{n=1}^{\infty}\frac{r^n}{n!}\,\mathrm{ad}^n_\Delta\big(\sigma\nabla_w - \overline{\sigma}\nabla_{\overline{w}}\big) =\\ &= u + \frac{i\sigma\big(e^{-2rN}-1\big)}{2N\pi}\nabla_w + \frac{i\overline{\sigma}\big(e^{2rN}-1\big)}{2N\pi}\nabla_{\overline{w}}\,.\end{aligned}$$

Similarly, the multiplication by v becomes

$$\begin{aligned}\exp\big(r\Delta\big)v\exp\big(-r\Delta\big) &= \sum_{n=0}^{\infty}\frac{r^n}{n!}\,\mathrm{ad}^n_\Delta(v) =\\ &= v - \frac{i}{\pi}\sum_{n=1}^{\infty}\frac{r^n}{n!}\,\mathrm{ad}^n_\Delta\big(\nabla_w - \nabla_{\overline{w}}\big) =\\ &= v + \frac{i\big(e^{-2rN}-1\big)}{2N\pi}\nabla_w + \frac{i\big(e^{2rN}-1\big)}{2N\pi}\nabla_{\overline{w}}\,.\end{aligned}$$

Since ∇_w is a $(-2N)$-eigenvector for ad_Δ, we simply have

$$\exp\big(r\Delta\big)\nabla_w\exp\big(-r\Delta\big) = e^{-2rN}\nabla_w\,.$$

Putting the pieces together we finally find

$$\begin{aligned}\widehat{U} &= u + \frac{i\sigma\big(e^{-2rN}-1\big)}{2N\pi}\nabla_w + \frac{i\overline{\sigma}\big(e^{2rN}-1\big)}{2N\pi}\nabla_{\overline{w}} - \frac{i\sigma e^{-2rN}}{\pi t}\nabla_w =\\ &= u - \frac{i\sigma\big(\overline{t}e^{-2rN}+t\big)}{2Nt\pi}\nabla_w + \frac{i\overline{\sigma}\big(e^{2rN}-1\big)}{2N\pi}\nabla_{\overline{w}} =\\ &= u + \frac{i\sigma\big(e^{2rN}-1\big)}{2N\pi}\nabla_w + \frac{i\overline{\sigma}\big(e^{2rN}-1\big)}{2N\pi}\nabla_{\overline{w}} = u - i\frac{e^{2rN}-1}{2N\pi}\nabla_v\,.\end{aligned}$$

Here we used the relation defining r, and that

$$\sigma\nabla_w + \overline{\sigma}\nabla_{\overline{w}} = \frac{1}{\sigma - \overline{\sigma}}\Big(-\sigma\overline{\sigma}\nabla_u - \sigma\nabla_v + \sigma\overline{\sigma}\nabla_u + \overline{\sigma}\nabla_v\Big) = -\nabla_v\,.$$

In complete analogy one has

$$\begin{aligned}\widehat{V} &= v + \frac{i\big(e^{-2rN}-1\big)}{2N\pi}\nabla_w + \frac{i\big(e^{2rN}-1\big)}{2N\pi}\nabla_{\overline{w}} - \frac{ie^{-2rN}}{\pi t}\nabla_w = \\ &= v - \frac{i\big(\overline{t}e^{-2rN}+t\big)}{2Nt\pi}\nabla_w + \frac{i\big(e^{2rN}-1\big)}{2N\pi}\nabla_{\overline{w}} = \\ &= v + \frac{i\big(e^{2rN}-1\big)}{2N\pi}\nabla_w + \frac{i\big(e^{2rN}-1\big)}{2N\pi}\nabla_{\overline{w}} = v + i\frac{e^{2rN}-1}{2N\pi}\nabla_u\,.\end{aligned}$$

In the last step we used that

$$\nabla_w + \nabla_{\overline{w}} = \frac{1}{\sigma - \overline{\sigma}}\Big(-\overline{\sigma}\nabla_u - \nabla_v + \sigma\nabla_u + \nabla_v\Big) = \nabla_u\,.$$

The relation for $\widehat{m}$ and $\widehat{\ell}$ follow from the fact that taking the exponential of an operator commutes with conjugating it by a unitary map. The splitting is a consequence of the Baker-Campbell-Hausdorff formula, which can be applied here since u commutes with ∇_v, and v with ∇_u. □

4.3 The Weil-Gel'fand-Zak transform

Lemma 11. *The spaces $\mathcal{C}^\infty(\mathbb{T}^2,\mathcal{L}^N)$ and $\mathcal{S}(\mathbb{A}_N,\mathbb{C})$ are isometric via the transformation $W^{(N)}:\mathcal{S}(\mathbb{A}_N,\mathbb{C})\to\mathcal{C}^\infty(\mathbb{T}^2,\mathcal{L}^N)$ expressed by*

$$f(x,n)\mapsto s(u,v) = e^{i\pi Nuv}\sum_{m\in\mathbb{Z}} f\Big(\sqrt{N}u + \frac{m}{\sqrt{N}}, -m\Big)e^{2\pi imv}\,.$$

Since this is an isometry, it extends in a natural way to the L^2*-completions of the two spaces.*

The above map is called the Weil-Gel'fand-Zak transform. We shall now study the conjugation of the quantum operators on $\mathcal{H}^{(t)}$ by the Weil-Gel'fand-Zak transform.

Lemma 12. *The following relations hold for every $f\in\mathcal{S}(\mathbb{A}_N,\mathbb{C})$*

$$\begin{aligned}\nabla_u W^{(N)}\big(f(\mathbf{x})\big) &= W^{(N)}\big(\sqrt{N}f'(\mathbf{x})\big)\,,\\ \nabla_v W^{(N)}\big(f(\mathbf{x})\big) &= W^{(N)}\Big(2\pi i\sqrt{N}xf(\mathbf{x})\Big)\,,\\ e^{2\pi iu}W^{(N)}\big(f(\mathbf{x})\big) &= W^{(N)}\Big(e^{2\pi i\frac{x}{\sqrt{N}}}e^{2\pi i\frac{n}{N}}f(\mathbf{x})\Big)\,,\\ e^{2\pi iv}W^{(N)}\big(f(\mathbf{x})\big) &= W^{(N)}\bigg(f\Big(x-\frac{1}{\sqrt{N}}, n+1\Big)\bigg)\,.\end{aligned}$$

Proof. This follows directly from computations, by applying the definitions. Notice that the derivatives commute with the infinite sum due to the properties of Schwartz class functions

$$\begin{aligned}\nabla_u W^{(N)}\big(f(\mathbf{x})\big) = \frac{\partial}{\partial u} W^{(N)}\big(f(\mathbf{x})\big) - i\pi N v W^{(N)}\big(f(\mathbf{x})\big) = \\ = e^{i\pi N uv} \sum_{m\in\mathbb{Z}} \frac{\partial}{\partial u} f\Big(\sqrt{N}u + \frac{m}{\sqrt{N}}, -m\Big) e^{2\pi i m v} + \\ + i\pi N v W^{(N)}\big(f\big) - i\pi N v W^{(N)}\big(f\big) = \\ = e^{i\pi N uv} \sum_{m\in\mathbb{Z}} \sqrt{N} f'\Big(\sqrt{N}u + \frac{m}{\sqrt{N}}, -m\Big) e^{2\pi i m v}\,.\end{aligned}$$

The second computation is similar

$$\begin{aligned}\nabla_v W^{(N)}\big(f(\mathbf{x})\big) = \frac{\partial}{\partial v} W^{(N)}\big(f(\mathbf{x})\big) + N\pi i u W^{(N)}\big(f(\mathbf{x})\big) = \\ = N\pi i u e^{N\pi i uv} \sum_{m\in\mathbb{Z}} f\Big(\sqrt{N}u + \frac{m}{\sqrt{N}}, -m\Big) e^{2\pi i m v} + \\ 2m\pi i e^{N\pi i uv} \sum_{m\in\mathbb{Z}} f\Big(\sqrt{N}u + \frac{m}{\sqrt{N}}, -m\Big) e^{2\pi i m v} + \\ + N\pi i u W^{(N)}\big(f(x,n)\big) = \\ = 2\pi i \sqrt{N} e^{N\pi i uv} \sum_{m\in\mathbb{Z}} \Big(\sqrt{N}u + \frac{m}{\sqrt{N}}\Big) f\Big(\sqrt{N}u + \frac{m}{\sqrt{N}}, -m\Big) e^{2\pi i m v}\,.\end{aligned}$$

The crucial passage in the other two computations is a change of variable in the sums

$$\begin{aligned}e^{2\pi i u} W^{(N)}\big(f(\mathbf{x})\big) = e^{i\pi N uv} \sum_{m\in\mathbb{Z}} e^{2\pi i u} f\Big(\sqrt{N}u + \frac{m}{\sqrt{N}}, -m\Big) e^{2\pi i m v} = \\ = e^{i\pi N uv} \sum_{m\in\mathbb{Z}} e^{2\pi i (u + \frac{m}{N})} e^{-2\pi i \frac{m}{N}} f\Big(\sqrt{N}u + \frac{m}{\sqrt{N}}, -m\Big) e^{2\pi i m v}\,.\end{aligned}$$

For the last relation we have that

$$\begin{aligned}e^{2\pi i v} W^{(N)}\big(f(\mathbf{x})\big) = e^{i\pi N uv} \sum_{m\in\mathbb{Z}} f\Big(\sqrt{N}u + \frac{m}{\sqrt{N}}, -m\Big) e^{2\pi i (m+1) v} = \\ = e^{N\pi i uv} \sum_{m\in\mathbb{Z}} f\Big(\sqrt{N}u + \frac{m-1}{\sqrt{N}}, -m+1\Big) e^{-2\pi i m v}\,.\end{aligned}$$

This concludes the proof. □

Proof of Theorem 2. We need to check that, for the given values of the quantum parameters, the Weil-Gel'fand-Zak transform induced the correspondence

$$\widehat{m} \longmapsto \widehat{m}_{\mathbf{x}}\,, \qquad \widehat{\ell} \longmapsto \widehat{\ell}_{\mathbf{x}}\,.$$

We proceed again by direct computation

$$\widehat{m}W^{(N)}\big(f(\mathbf{x})\big) = e^{2\pi i u} \exp\left(\frac{e^{2Nr}-1}{N}\nabla_v\right) W^{(N)}\big(f(\mathbf{x})\big) =$$

$$= e^{2\pi i u} W^{(N)}\left(\exp\left(2\pi i \frac{e^{2Nr}-1}{\sqrt{N}} x\right) f(\mathbf{x})\right) =$$

$$= W^{(N)}\left(\exp\left(2\pi i \frac{e^{2Nr}}{\sqrt{N}} x\right) e^{2\pi i \frac{n}{N}} f(\mathbf{x})\right).$$

For the next computation we shall use the fact that, if λ is a complex parameter, the exponential of $\lambda \frac{\mathrm{d}}{\mathrm{d}x}$ in $\mathrm{L}^2(\mathbb{R},\mathbb{C})$ is the shift by λ, as discussed immediately after Definition 3

$$\widehat{\ell}W^{(N)}\big(f(\mathbf{x})\big) = e^{2\pi i v} \exp\left(-\frac{e^{2Nr}-1}{N}\nabla_u\right) W^{(N)}\big(f(\mathbf{x})\big) =$$

$$= e^{2\pi i v} W^{(N)}\left(f\left(x - \frac{e^{2Nr}-1}{\sqrt{N}}, n\right)\right) =$$

$$= W^{(N)}\left(f\left(x - \frac{e^{2Nr}}{\sqrt{N}}, n+1\right)\right).$$

As for the q-commutativity, the following relation can easily be checked directly from the expression of the operators, or deduced from Dirac's relation

$$\big[\widehat{U}, \widehat{V}\big] = -i\{U, V\} = -i \cdot \frac{-1}{\pi t} = \frac{i}{\pi t}.$$

The result follows from this and the Baker-Campbell-Hausdorff formula. □

5 Operators annihilating the Andersen-Kashaev invariant

Throughout this section we will always assume that N is an *odd* positive integer.

For a fixed level $t = N + iS$ and b and q as above there is an action of the algebra $\mathcal{A}_{\mathrm{loc}}$ from (4) on the space of meromorphic functions on $\mathbb{A}_N^{\mathbb{C}}$ by

$$Q \mapsto \widehat{m}_{\mathbf{x}}, \qquad E \mapsto \widehat{\ell}_{\mathbf{x}}.$$

As before, if f is a meromorphic function it makes sense to consider its annihilating left ideals $\mathcal{I}(f)$ and $\mathcal{I}_{\mathrm{loc}}(f)$ in $\mathcal{A}_{\mathrm{loc}}$ and $\mathcal{A}$, respectively:

$$\mathcal{I}_{\mathrm{loc}}(f) = \left\{\, p \in \mathcal{A}_{\mathrm{loc}} : p(\widehat{m}_{\mathbf{x}}, \widehat{\ell}_{\mathbf{x}}) f = 0 \,\right\}, \qquad \mathcal{I}(f) = \mathcal{I}_{\mathrm{loc}}(f) \cap \mathcal{A}.$$

Notice moreover that, if p_1 is a polynomial in $\widehat{m}_{\mathbf{x}}$ alone and j a non-negative integer, then

$$p_1 \widehat{\ell}_{\mathbf{x}}^{\,j} p_2 \in \mathcal{I}(f) \implies p_2 \in \mathcal{I}(f). \tag{17}$$

We extend the notation to the case of a function f on $\mathbb{A}_N$ having a unique meromorphic extension to $\mathbb{A}_N^{\mathbb{C}}$.

Definition 6. *Suppose that* $f = J^{(\mathrm{b},N)}_{M,K}$ *for a hyperbolic knot* K *in a closed, oriented compact 3-manifold* M*, for which Conjecture 3 holds. We define the non-commutative polynomial* $\widehat{A}^{\mathbb{C}}_{t,(M,K)}$ *as the unique element of* $\mathcal{I}_{loc}(J^{(\mathrm{b},N)}_{M,K})$ *of lowest degree in* $\widehat{\ell}_{\mathbf{x}}$ *with integral and co-prime coefficients.*

We are now ready to rephrase Theorem 3 more precisely.

Theorem 13. *For* $K \subseteq S^3$ *the figure-eight knot* 4_1 *or* 5_2*, we have*

$$\widehat{A}^{\mathbb{C}}_{t,K}(\widehat{m}_{\mathbf{x}}, \widehat{\ell}_{\mathbf{x}}) \cdot \left(\widehat{m}_{\mathbf{x}} - 1\right) = \widehat{A}_{q,K}(\widehat{m}_{\mathbf{x}}, \widehat{\ell}_{\mathbf{x}}) \,.$$

In the semi-classical limit $t \to \infty$*, we have that*

$$\left(m^4 - 1\right) \cdot \widehat{A}^{\mathbb{C}}_{\infty,K}(m^2, \ell) = A_K(m, \ell) \,.$$

We shall dedicate the rest of the paper to the proof of this statement.

5.1 The figure-eight knot 4_1

5.1.1 The invariant and its holomorphic extension

Recall the formula for $\chi^{(\mathrm{b},N)}_{4_1}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{A}_N \subseteq \mathbb{C} \times \mathbb{Z}/N\mathbb{Z}$ (see e.g. [AM16b])

$$\chi^{(\mathrm{b},N)}_{4_1}(\mathbf{x}) := \int_{\mathbb{A}_N} \frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\langle \mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\langle \mathbf{y}\rangle^{-2}} \, d\mathbf{y} \,. \tag{18}$$

We shall often omit the superscript (b, N). It follows from lemma 4 that, for $\mathbf{x}$ and $\mathbf{y}$ real, the integrand has constant absolute value 1, so the integral is not absolutely convergent. However, suppose $y = \eta + i\varepsilon\frac{\mathrm{b}}{\sqrt{N}}$ for real η and ε. One may write

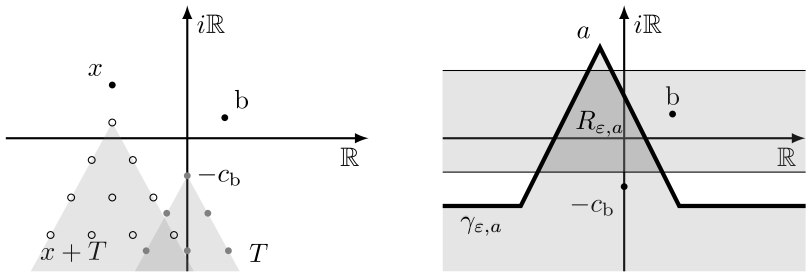
$$\left|\langle \mathbf{y}\rangle\right| = e^{-\pi\,\mathrm{Im}(y^2)} = e^{\varepsilon^2\frac{\mathrm{Im}(\mathrm{b}^2)}{N}} e^{-2\pi\eta\varepsilon\frac{\mathrm{Re}(\mathrm{b})}{\sqrt{N}}}$$
$$\left|\langle \mathbf{x}-\mathbf{y}\rangle\right| = e^{-\pi\,\mathrm{Im}((x-y)^2)} = e^{\varepsilon^2\frac{\mathrm{Im}(\mathrm{b}^2)}{N}} e^{-2\pi(\eta-x)\varepsilon\frac{\mathrm{Re}(\mathrm{b})}{\sqrt{N}}}$$

Then the same lemma shows that, for appropriate smooth functions C_- and C_+ of $\mathbf{x}$ and ε alone, one has

$$\left|\frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\,\langle \mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\,\langle \mathbf{y}\rangle^{-2}}\right| \approx \begin{cases} \left|\dfrac{\langle \mathbf{x}-\mathbf{y}\rangle^{-1}}{\langle \mathbf{y}\rangle^{-2}}\right| = C_-(\mathbf{x},\varepsilon)e^{-2\pi\eta\varepsilon\frac{\mathrm{Re}(\mathrm{b})}{\sqrt{N}}} & \eta \to -\infty\,, \\ \left|\dfrac{\langle \mathbf{x}-\mathbf{y}\rangle^{-2}}{\langle \mathbf{y}\rangle^{-1}}\right| = C_+(\mathbf{x},\varepsilon)e^{4\pi\eta\varepsilon\frac{\mathrm{Re}(\mathrm{b})}{\sqrt{N}}} & \eta \to +\infty\,. \end{cases}$$

Therefore, the integrand decays exponentially in η for every fixed $\varepsilon < 0$, and the integral converges absolutely on $\mathbb{A}_N + i\varepsilon\mathrm{b}/\sqrt{N}$, provided that this contour does not cross any poles.

Recall that the zeroes of $\varphi_{\mathrm{b}}(\mathbf{y})$ only occur for y in the lower infinite triangle T, and similarly the poles of $\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})$ have $y - x \in T$, or equivalently $y \in T + x$. Therefore, since $c_{\mathrm{b}} = i\,\mathrm{Re}(\mathrm{b})$, the integral over the contour $\mathbb{A}_N + i\varepsilon\mathrm{b}/\sqrt{N}$ is absolutely convergent for $-1 < \varepsilon < 0$. Moreover, given the exponential decay of the integrand at infinity, the residue theorem and dominated convergence



(a) For fixed (x, n) and m, the poles of the integrand occur inside the infinite triangles.

(b) The integrand decays exponentially for x in the strip, and the poles lie below $\gamma_{\varepsilon,a}$ if $-c_{\mathrm{b}}$ and x do.

Figure 1: The distribution of the poles of the integrand and the contour are illustrated for $N = 1$. The situation is analogous for higher N, up to rescaling c_{b} by $\sqrt{N}$ and replicating the picture N times.

together show that, in the limit for $\varepsilon \to 0$, one recovers the improper integral over $\mathbb{A}_N$. In fact, the same value is obtained for each $\mathbf{x}$ if the contour is pushed up within a compact region of $\mathbb{C} \times \mathbb{Z}/N\mathbb{Z}$. In other words, one may use *any* contour Γ which goes along $\mathbb{A}_N + i\varepsilon \mathrm{b}/\sqrt{N}$ near ∞, provided that

- $\varepsilon < 0$,
- all the poles of the integrand lie below Γ.

In order to study the action of the operators on χ_{4_1}, and in particular the shift, we need to understand its holomorphic extension to $\mathbb{C} \times \mathbb{Z}/N\mathbb{Z}$. To this end, we shall fix a contour Γ as above and determine for which values of $\mathbf{x}$ the integral converges absolutely. If $x = \xi + i\lambda \mathrm{b}/\sqrt{N}$ and $y = \eta + i\varepsilon/\sqrt{N}$ for $\xi, \eta, \varepsilon, \lambda \in \mathbb{R}$, then one has

$$\left|\langle \mathbf{x} - \mathbf{y} \rangle\right| = e^{-\pi \operatorname{Im}((x-y)^2)} = e^{(\lambda-\varepsilon)^2 \operatorname{Im}(\mathrm{b}^2)} e^{-2\pi(\eta-\xi)(\varepsilon-\lambda)\frac{\operatorname{Re}(\mathrm{b})}{\sqrt{N}}} .$$

The behaviour of the integrand at infinity is then given by

$$\left| \frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y}) \langle \mathbf{x}-\mathbf{y} \rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y}) \langle \mathbf{y} \rangle^{-2}} \right| \approx \begin{cases} \dfrac{\langle \mathbf{x}-\mathbf{y} \rangle^{-1}}{\langle \mathbf{y} \rangle^{-2}} = C_{-}(x,\varepsilon) e^{-2\pi\eta(\varepsilon+\lambda)\frac{\operatorname{Re}(\mathrm{b})}{\sqrt{N}}} & \eta \to -\infty \,, \\ \dfrac{\langle \mathbf{x}-\mathbf{y} \rangle^{-2}}{\langle \mathbf{y} \rangle^{-1}} = C_{+}(x,\varepsilon) e^{2\pi\eta(\varepsilon-2\lambda)\frac{\operatorname{Re}(\mathrm{b})}{\sqrt{N}}} & \eta \to +\infty \,. \end{cases}$$

Therefore, the convergence condition near ∞ is equivalent to

$$\frac{\varepsilon}{2} < \lambda < -\varepsilon \,.$$

Fix now $\varepsilon < 0$ and $a \in -2c_{\mathrm{b}}/\sqrt{N} - T$, and consider the contour $\gamma_{\varepsilon,a}$ in $\mathbb{C}$ which deviates from $\mathbb{R} + i\varepsilon\mathrm{b}/\sqrt{N}$ along $a + i\mathbb{R}\overline{\mathrm{b}}$ and $a + i\mathbb{R}\mathrm{b}$. The condition on a ensures that the tip of T, and hence the whole triangle, lies below this contour. Similarly, $\gamma_{\varepsilon,a}$ avoids $T + x$ if and only if it stays above its tip $x - N^{1/2}c_{\mathrm{b}}$, which is the case if e.g. $x \in T + a$. If Γ is chosen to be $\gamma_{\varepsilon,a}$ in each component of

$\mathbb{C}\times\mathbb{Z}/N\mathbb{Z}$, the integral converges absolutely and defines a holomorphic function at least on the region $R_{\varepsilon,a}\times\mathbb{Z}/N\mathbb{Z}$, where

$$R_{\varepsilon,a} := \left\{ x=\xi+i\lambda\frac{\mathrm{b}}{\sqrt{N}}\in T+a\,,\frac{\varepsilon}{2}<\lambda<-\varepsilon \right\}.$$

The resulting function agrees with the one defined by the improper integral over $\mathbb{A}_N$ on $R_{\varepsilon,a}\cap\mathbb{R}$, which is non-empty since it always contains a neighbourhood of 0. Also, as ε and a vary on the allowed domains, these regions cover the whole complex plane, thus giving the full holomorphic extension of χ_{4_1} to $\mathbb{C}\times\mathbb{Z}/N\mathbb{Z}$.

5.1.2 Operators annihilating the integrand

We now approach the problem of studying the operators annihilating the integrand of (18). Following the plan illustrated above, we use Lemma 5 to see how $\widehat{\ell}_{\mathbf{x}}$ acts on it

$$\Big(1+q^{-\frac{1}{2}}\widehat{m}_{\mathbf{x}}^{-1}\widehat{m}_{\mathbf{y}}\Big)q\widehat{m}_{\mathbf{x}}^{2}\widehat{m}_{\mathbf{y}}^{-2}\frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\langle\mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\langle\mathbf{y}\rangle^{-2}}\,.$$

A simple manipulation translates this into

$$\left(\widehat{m}_{\mathbf{y}}^{2}\widehat{\ell}_{\mathbf{x}}-q^{\frac{1}{2}}\Big(q^{\frac{1}{2}}\widehat{m}_{\mathbf{x}}+\widehat{m}_{\mathbf{y}}\Big)\widehat{m}_{\mathbf{x}}\right)\frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\langle\mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\langle\mathbf{y}\rangle^{-2}}=0\,.$$

Similarly, the action of $\widehat{\ell}_{\mathbf{y}}$ gives

$$\Big(1+q^{\frac{1}{2}}\widehat{m}_{\mathbf{x}}^{-1}\widehat{m}_{\mathbf{y}}\Big)^{-1}\Big(1+q^{-\frac{1}{2}}\widehat{m}_{\mathbf{y}}^{-1}\Big)^{-1}\widehat{m}_{\mathbf{x}}^{-2}\frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\langle\mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\langle\mathbf{y}\rangle^{-2}}\,,$$

hence

$$\left(\Big(\widehat{m}_{\mathbf{x}}+q^{\frac{1}{2}}\widehat{m}_{\mathbf{y}}\Big)\Big(q^{\frac{1}{2}}\widehat{m}_{\mathbf{y}}+1\Big)\widehat{m}_{\mathbf{x}}\widehat{\ell}_{\mathbf{y}}-q^{\frac{1}{2}}\widehat{m}_{\mathbf{y}}\right)\frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\langle\mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\langle\mathbf{y}\rangle^{-2}}=0\,.$$

We can then start elimination in $\widehat{m}_{\mathbf{y}}$ on

$$\begin{aligned} g_1 &= \widehat{\ell}_{\mathbf{x}}\widehat{m}_{\mathbf{y}}^{2}-q^{\frac{1}{2}}\widehat{m}_{\mathbf{x}}\widehat{m}_{\mathbf{y}}-q\widehat{m}_{\mathbf{x}}^{2}\,,\\ g_2 &= \widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}\widehat{m}_{\mathbf{y}}^{2}+q^{\frac{1}{2}}\big(\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2}+\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}-q\big)\widehat{m}_{\mathbf{y}}+q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2}\,. \end{aligned}$$

We run the operation on a computer, and find

$$q^{\frac{9}{2}}\widehat{m}_{\mathbf{x}}^{2}\widehat{A}=qa_1g_1-a_2g_2\,,$$

where a_1 is given by

$$\begin{aligned} &\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}\Big(\big(q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2}-1\big)\big(q^{3}\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2}+q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}-1\big)\widehat{\ell}_{\mathbf{x}}+q^{2}\widehat{\ell}_{\mathbf{y}}\big(q^{3}\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2}-1\big)\widehat{m}_{\mathbf{x}}^{2}\Big)\widehat{m}_{\mathbf{y}}\\ &+q^{\frac{1}{2}}\big(q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2}-1\big)\\ &\cdot\Big(q^{5}\widehat{\ell}_{\mathbf{y}}^{2}\widehat{m}_{\mathbf{x}}^{4}+q^{3}\widehat{\ell}_{\mathbf{y}}^{2}\widehat{m}_{\mathbf{x}}^{3}+q\widehat{\ell}_{\mathbf{y}}^{2}\widehat{m}_{\mathbf{x}}^{2}-q^{2}(q+1)\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2}-(q+1)\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}+1\Big)\widehat{\ell}_{\mathbf{x}}\\ &+q^{\frac{5}{2}}\widehat{\ell}_{\mathbf{y}}\big(\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}-q\big)\big(q^{3}\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^{2}-1\big)\widehat{m}_{\mathbf{x}}^{2}\,, \end{aligned}$$

and a_2 is

$$\Big(\big(q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 - 1\big)\Big(q^3\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 + q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}} - 1\Big)\widehat{\ell}_{\mathbf{x}}^2 + q^3\widehat{\ell}_{\mathbf{y}}\Big(q^3\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 - 1\Big)\widehat{m}_{\mathbf{x}}^2\widehat{\ell}_{\mathbf{x}}\Big)\widehat{m}_{\mathbf{y}}$$
$$-q^{\frac{5}{2}}\widehat{\ell}_{\mathbf{y}}\big(q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 - 1\big)\widehat{m}_{\mathbf{x}}^2\widehat{\ell}_{\mathbf{x}}^2$$
$$-q^{\frac{3}{2}}\Big(q^4(q+1)\widehat{\ell}_{\mathbf{y}}^2\widehat{m}_{\mathbf{x}}^4 + q^2\widehat{\ell}_{\mathbf{y}}^2\widehat{m}_{\mathbf{x}}^3 - q(q^2+q+1)\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 - q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}} + 1\Big)\widehat{m}_{\mathbf{x}}\widehat{\ell}_{\mathbf{x}}$$
$$-q^{\frac{7}{2}}\widehat{\ell}_{\mathbf{y}}\big(q^3\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 - 1\big)\widehat{m}_{\mathbf{x}}^3$$

where

$$\begin{aligned}\widehat{A} = {} & q^3\widehat{\ell}_{\mathbf{y}}^2\big(q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 - 1\big)\widehat{m}_{\mathbf{x}}^2\widehat{\ell}_{\mathbf{x}}^2 \\ & - \big(q^2\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 - 1\big)\Big(q^4\widehat{\ell}_{\mathbf{y}}^2\widehat{m}_{\mathbf{x}}^4 - q^3\widehat{\ell}_{\mathbf{y}}^2\widehat{m}_{\mathbf{x}}^3 - q(q^2+1)\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 - q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}} + 1\Big)\widehat{\ell}_{\mathbf{x}} \\ & + q\widehat{\ell}_{\mathbf{y}}\big(q^3\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 - 1\big)\widehat{m}_{\mathbf{x}}^2 .\end{aligned} \tag{19}$$

It is apparent from this, (5) and (7) that this non-commutative polynomial, evaluated at $\widehat{\ell}_{\mathbf{y}} = 1$, verifies the statement of Theorem 13 up to rescaling $\widehat{\ell}_{\mathbf{x}}$ by q, and an overall factor q.

Theorem 14. *The complex non-commutative $\widehat{A}^{\mathbb{C}}$-polynomial for the figure-eight knot is given by the evaluation*

$$\widehat{A}^{\mathbb{C}}_{t,4_1}(\widehat{m}_{\mathbf{x}}, \widehat{\ell}_{\mathbf{x}}) = q^{-1}\widehat{A}(\widehat{m}_{\mathbf{x}}, q^{-1}\widehat{\ell}_{\mathbf{x}}, \widehat{\ell}_{\mathbf{y}} = 1) .$$

Proof. Since $q^{\frac{9}{2}}\widehat{m}_{\mathbf{x}}^2\widehat{A}$ is a left combination of g_1 and g_2, it annihilates the integrand defining χ_{4_1}, hence so does $\widehat{A}$ itself. For ε and a as usual, $\rho < \varepsilon$ a non-negative integer, call

$$\Gamma^{(\rho)}_{\varepsilon,a} := \Gamma_{\varepsilon,a} - i\rho\mathrm{b}/\sqrt{N} = \Gamma_{\varepsilon-\rho, a - i\frac{\rho\mathrm{b}}{\sqrt{N}}} .$$

Of course $\widehat{m}_{\mathbf{x}}$ and $\widehat{\ell}_{\mathbf{x}}$ commute with taking integrals in $\mathrm{d}\mathbf{y}$, so for any λ, μ and every $\mathbf{x} \in \mathbb{A}_N^{\mathbb{C}}$ one may write

$$\int_{\Gamma_{\varepsilon,a}} \widehat{\ell}_{\mathbf{y}}^{\rho}\widehat{m}_{\mathbf{x}}^{\mu}\widehat{\ell}_{\mathbf{x}}^{\lambda} \frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\langle \mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\langle \mathbf{y}\rangle^{-2}} \,\mathrm{d}\mathbf{y} = \widehat{m}_{\mathbf{x}}^{\mu}\widehat{\ell}_{\mathbf{x}}^{\lambda} \int_{\Gamma^{(\rho)}_{\varepsilon,a}} \frac{\varphi_{\mathrm{b}}(\mathbf{x}-\mathbf{y})\langle \mathbf{x}-\mathbf{y}\rangle^{-2}}{\varphi_{\mathrm{b}}(\mathbf{y})\langle \mathbf{y}\rangle^{-2}} \,\mathrm{d}\mathbf{y} .$$

The right-hand side gives $\widehat{m}_{\mathbf{x}}^{\mu}\widehat{\ell}_{\mathbf{x}}^{\lambda}\chi_{4_1}(\mathbf{x})$ for

$$\mathbf{x} \in R_{\varepsilon-\rho, a-i\frac{\rho\mathrm{b}}{\sqrt{N}}} + \left(i\frac{\lambda\mathrm{b}}{\sqrt{N}}, -1\right) .$$

If ε and $|a|$ are big enough, the intersection of these domains as $0 \le \rho \le 3$ and $0 \le \lambda \le 2$ is non-empty, and in fact these sets cover all $\mathbb{A}_N^{\mathbb{C}}$ for ε and a going to ∞. Applying this point-wise in $\mathbf{x}$ and monomial by monomial in $\widehat{A}$, this shows that this polynomial, evaluated at $\widehat{\ell}_{\mathbf{y}} = 1$, annihilates χ_{4_1}.

On the other hand, recall that

$$J^{(\mathrm{b},N)}_{S^3,4_1}(\mathbf{x}) = e^{4\pi i\frac{c_{\mathrm{b}}x}{\sqrt{N}}}\chi_{4_1}(\mathbf{x}) .$$

As is easily checked, this implies that $\widehat{A}(\widehat{m}_{\mathbf{x}}, q^{-1}\widehat{\ell}_{\mathbf{x}}, 1)$ belongs to $\mathcal{I}_{\mathrm{loc}}(4_1)$, so there exists $p \in \mathcal{A}_{\mathrm{loc}}$ such that

$$\widehat{A}(\widehat{m}_{\mathbf{x}}, q^{-1}\widehat{\ell}_{\mathbf{x}}, 1) = p(\widehat{m}_{\mathbf{x}}, \widehat{\ell}_{\mathbf{x}}) \cdot \widehat{A}^{\mathbb{C}}_{t,4_1}(\widehat{m}_{\mathbf{x}}, \widehat{\ell}_{\mathbf{x}}) .$$

However, for $q = 1$ the left-hand side gives $(m^4 - 1)A_{4_1}$, so the expression on the right-hand side gives a factorisation of this polynomial. On the other hand, the classical A-polynomial is irreducible, which implies that either p or $\widehat{A}^{\mathbb{C}}_{t,4_1}$ is a polynomial in $\widehat{m}_{\mathbf{x}}$ alone. If this were the case for $\widehat{A}^{\mathbb{C}}_{t,4_1}$, this would mean that $\chi_{4_1} = 0$, which is not the case as, for instance, this would contradict its known asymptotic properties. This means that $q^{-1}\widehat{A}(\widehat{m}_{\mathbf{x}}, q^{-1}\widehat{\ell}_{\mathbf{x}}, 1)$ is proportional to $\widehat{A}^{\mathbb{C}}_{t,4_1}$, and since its coefficients satisfy the required conditions the conclusion follows. □

This completes the argument for the figure-eight knot.

5.2 The knot 5_2

For the knot 5_2, the function $\chi^{(N)}_{5_2} = \chi_{5_2}$ on $\mathbb{A}_N$ is given by the integral

$$\chi_{5_2}(\mathbf{x}) := \int_{\mathbb{A}_N} \frac{\langle \mathbf{y} \rangle \langle \mathbf{x} \rangle^{-1}}{\varphi_{\mathrm{b}}(\mathbf{y}+\mathbf{x})\varphi_{\mathrm{b}}(\mathbf{y})\varphi_{\mathrm{b}}(\mathbf{y}-\mathbf{x})} \, d\mathbf{y} . \tag{20}$$

As in the case of the figure-eight knot, we need first of all to discuss the convergence of the integral. Calling $\Phi(\mathbf{x}, \mathbf{y})$ the integrand, for $\mathbf{x}$ fixed and $\mathbf{y}$ of the usual form we have the following asymptotic behaviour:

$$\left|\Phi(\mathbf{x},\mathbf{y})\right| \approx \begin{cases} \left|\langle \mathbf{y} \rangle \langle \mathbf{x} \rangle^{-1}\right| = C_-(\mathbf{x},\varepsilon) e^{-2\pi\eta\varepsilon\frac{\mathrm{Re(b)}}{\sqrt{N}}} & \eta \to -\infty , \\ \left|\frac{1}{\langle \mathbf{y}+\mathbf{x} \rangle \langle \mathbf{y}-\mathbf{x} \rangle}\right| = \left|\frac{1}{\langle \mathbf{x} \rangle^2 \langle \mathbf{y} \rangle^2}\right| = C_+(\mathbf{x},\varepsilon) e^{4\pi\eta\varepsilon\frac{\mathrm{Re(b)}}{\sqrt{N}}} & \eta \to +\infty . \end{cases}$$

Independently on $\mathbf{x}$, the integrand decays exponentially as $\eta \to \infty$ as long as ε is positive. Moreover, for fixed $\mathbf{x}$ the poles of Φ lie inside of the region

$$T \cup (T + \mathbf{x}) \cup (T - \mathbf{x}) .$$

Using $\Gamma_{\varepsilon,a}$ as above, the integral is absolutely convergent if e.g. both $\mathbf{x}$ and $-\mathbf{x}$ lie below the contour, which is to say that $\mathbf{x}$ lies below $\Gamma_{\varepsilon,a}$ and above $-\Gamma_{\varepsilon,a}$. This defines a holomorphic function in the region R_a delimited by the four lines $\pm a + i\mathbb{R}\mathrm{b}$ and $\pm a + i\mathbb{R}\overline{\mathrm{b}}$

$$R_a := \left\{ (x,n) : x \in \left(a + \frac{c_{\mathrm{b}}}{\sqrt{N}} + T\right) \cap \left(-a - \frac{c_{\mathrm{b}}}{\sqrt{N}} - T\right) \right\}$$

Again, by a combination of the residue theorem and dominated convergence, the result is independent on the choice of the path, and the limit $\varepsilon \to 0$ with $a = 0$ gives the improper integral along $\mathbb{A}_N$ for $\mathbf{x}$ real. This proves that χ_{5_2} can be extended to a holomorphic function over the whole $\mathbb{C} \times \mathbb{Z}/N\mathbb{Z}$. We want to stress that, since R_a depends on a alone, the specific choice of $\varepsilon < 0$ has no influence on the region of convergence of the integral.

5.2.1 Operators annihilating the integrand

Following the lines of the case of the figure-eight knot, we use Lemma 5 to determine the action of $\widehat{\ell}_{\mathbf{x}}$ and $\widehat{\ell}_{\mathbf{y}}$ on the integrand. The first operator acts as

$$q^{\frac{1}{2}}\widehat{m}_{\mathbf{x}}\big(1+q^{-\frac{1}{2}}\widehat{m}_{\mathbf{x}}^{-1}\widehat{m}_{\mathbf{y}}^{-1}\big)^{-1}\big(1+q^{\frac{1}{2}}\widehat{m}_{\mathbf{x}}\widehat{m}_{\mathbf{y}}^{-1}\big)\,,$$

while the second gives

$$q^{-\frac{1}{2}}\widehat{m}_{\mathbf{y}}^{-1}\big(1+q^{-\frac{1}{2}}\widehat{m}_{\mathbf{x}}^{-1}\widehat{m}_{\mathbf{y}}^{-1}\big)^{-1}\big(1+q^{-\frac{1}{2}}\widehat{m}_{\mathbf{y}}^{-1}\big)^{-1}\big(1+q^{-\frac{1}{2}}\widehat{m}_{\mathbf{y}}^{-1}\widehat{m}_{\mathbf{x}}\big)^{-1}\,.$$

From this we get the operators

$$\begin{aligned}
g_1 &= q^{\frac{1}{2}}\big(\widehat{m}_{\mathbf{x}}\widehat{\ell}_{\mathbf{x}}-q^{\frac{1}{2}}\widehat{m}_{\mathbf{x}}^2\big)\widehat{m}_{\mathbf{y}}+\widehat{\ell}_{\mathbf{x}}-q^{\frac{3}{2}}\widehat{m}_{\mathbf{x}}^3\,,\\
g_2 &= \widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}\widehat{m}_{\mathbf{y}}^3+q^{\frac{1}{2}}\Big(\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2+\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}-q^2\widehat{m}_{\mathbf{x}}+\widehat{\ell}_{\mathbf{y}}\Big)\widehat{m}_{\mathbf{y}}^2\\
&\quad+q\widehat{\ell}_{\mathbf{y}}\Big(\widehat{m}_{\mathbf{x}}^2+\widehat{m}_{\mathbf{x}}+1\Big)\widehat{m}_{\mathbf{y}}+q^{\frac{3}{2}}\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}\,.
\end{aligned}$$

Again, running elimination in $\widehat{m}_{\mathbf{y}}$ returns

$$\widehat{A}=a_1g_1+\widehat{m}_{\mathbf{x}}g_2\,,$$

where a_1 is expressed by

$$\begin{aligned}
&\Big(-q^{\frac{1}{2}}\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}\big(q\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^2\widehat{m}_{\mathbf{x}}^2-1\big)\widehat{\ell}_{\mathbf{x}}^2+q^2(q+1)\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2\big(q\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^5\widehat{m}_{\mathbf{x}}^2-1\big)\widehat{\ell}_{\mathbf{x}}\\
&-q^{\frac{7}{2}}\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^3\big(q^4\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^5\widehat{m}_{\mathbf{x}}^2-1\big)\Big)\widehat{m}_{\mathbf{y}}^2\\
&+\Big(-q\widehat{m}_{\mathbf{x}}\big(q\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^2\widehat{m}_{\mathbf{x}}-1\big)\Big(q^3\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}+\widehat{\ell}_{\mathbf{y}}-q^2\Big)\widehat{\ell}_{\mathbf{x}}^2\\
&+q^{\frac{5}{2}}\widehat{m}_{\mathbf{x}}\big(q\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^5\widehat{m}_{\mathbf{x}}^2-1\big)\Big(q^3\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2+(q+1)\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}-q^2(q+1)\widehat{m}_{\mathbf{x}}+\widehat{\ell}_{\mathbf{y}}\Big)\widehat{\ell}_{\mathbf{x}}\\
&-q^8\widehat{m}_{\mathbf{x}}^2\big(q^4\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^5\widehat{m}_{\mathbf{x}}^2-1\big)\Big(\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}-q^2\widehat{m}_{\mathbf{x}}+\widehat{\ell}_{\mathbf{y}}\Big)\Big)\widehat{m}_{\mathbf{y}}\\
&-q^{\frac{1}{2}}\big(q\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^2\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^4\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2+1\big)\widehat{\ell}_{\mathbf{x}}^2-q\big(q\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^5\widehat{m}_{\mathbf{x}}^2-1\big)\\
&\cdot\Big(q^6\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^4-q^5\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^3-q^5\widehat{m}_{\mathbf{x}}^3-q^2(q^2+1)\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2-q^2\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}-q^2\widehat{m}_{\mathbf{x}}+\widehat{\ell}_{\mathbf{y}}\Big)\widehat{\ell}_{\mathbf{x}}\\
&-q^{\frac{9}{2}}\widehat{m}_{\mathbf{x}}^2\big(q^2\widehat{m}_{\mathbf{x}}^2+\widehat{\ell}_{\mathbf{y}}\big)\big(q^4\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^5\widehat{m}_{\mathbf{x}}^2-1\big)\,,
\end{aligned}$$

and a_2 is

$$\begin{aligned}
&\big(q\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^2\widehat{m}_{\mathbf{x}}^2-1\big)\widehat{\ell}_{\mathbf{x}}^3-q^{\frac{3}{2}}\widehat{m}_{\mathbf{x}}\big(q^2+q+1\big)\big(q\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^4\widehat{m}_{\mathbf{x}}^2-1\big)\widehat{\ell}_{\mathbf{x}}^2\\
&+q^3\widehat{m}_{\mathbf{x}}^2\big(q^2+q+1\big)\big(q^2\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^5\widehat{m}_{\mathbf{x}}^2-1\big)\widehat{\ell}_{\mathbf{x}}\\
&-q^{\frac{9}{2}}\widehat{m}_{\mathbf{x}}^3\big(q^4\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^5\widehat{m}_{\mathbf{x}}^2-1\big)\,,
\end{aligned}$$

where

$$
\begin{aligned}
\widehat{A} = & -q^{\frac{1}{2}}\big(q\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^2\widehat{m}_{\mathbf{x}}^2-1\big)\widehat{\ell}_{\mathbf{x}}^3 \\
& + q\big(q\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^4\widehat{m}_{\mathbf{x}}^2-1\big)\Big(q^9\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^5 - q^7\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^4 - q^4(q^3+1)\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^3 \\
& + q^5(q+1)\widehat{m}_{\mathbf{x}}^3 + q^2(q^3+1)\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 + q^2(\widehat{\ell}_{\mathbf{y}}+1)\widehat{m}_{\mathbf{x}} - \widehat{\ell}_{\mathbf{y}}\Big)\widehat{\ell}_{\mathbf{x}}^2 \\
& + q^{\frac{9}{2}}\widehat{m}_{\mathbf{x}}^2\big(q^2\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^5\widehat{m}_{\mathbf{x}}^2-1\big)\Big(q^6\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^5 - q^5(\widehat{\ell}_{\mathbf{y}}+1)\widehat{m}_{\mathbf{x}}^4 \\
& - q^2(q^3+1)\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^3 + q(q^3\widehat{\ell}_{\mathbf{y}} - q^2 - q + \widehat{\ell}_{\mathbf{y}})\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}}^2 + q\widehat{\ell}_{\mathbf{y}}\widehat{m}_{\mathbf{x}} - \widehat{\ell}_{\mathbf{y}}\Big)\widehat{\ell}_{\mathbf{x}} \\
& + q^8\widehat{m}_{\mathbf{x}}^7\big(q^4\widehat{m}_{\mathbf{x}}^2-1\big)\big(q^5\widehat{m}_{\mathbf{x}}^2-1\big)\,.
\end{aligned}
$$

As for the figure-eight knot, the evaluation at $\widehat{\ell}_{\mathbf{y}} = 1$ yields a non-commutative polynomial in $\widehat{m}_{\mathbf{x}}$ and $\widehat{\ell}_{\mathbf{x}}$ which, up to a global factor in q and a rescaling of $\widehat{\ell}_{\mathbf{x}}$ by $-q^{\frac{1}{2}}$, verifies Theorem 13. With few slight adaptations, the same argument as for the previous case finally shows the following.

Theorem 15. *The complex non-commutative $\widehat{A}^{\mathbb{C}}$-polynomial for the knot 5_2 is given by the evaluation*

$$
\widehat{A}^{\mathbb{C}}_{t,4_1}(\widehat{m}_{\mathbf{x}}, \widehat{\ell}_{\mathbf{x}}) = q^{-1}\widehat{A}(\widehat{m}_{\mathbf{x}}, -q^{-\frac{1}{2}}\widehat{\ell}_{\mathbf{x}}, \widehat{\ell}_{\mathbf{y}} = 1)\,.
$$

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