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Geometric recursion

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Abstract

We propose a general theory whose main component are functorial assignments

$$\mathbf{\Sigma} \mapsto \Omega_{\mathbf{\Sigma}} \in \mathbf{E}(\mathbf{\Sigma}),$$

for a large class of functors $\mathbf{E}$ from a certain category of bordered surfaces ($\mathbf{\Sigma}$'s) to a suitable a target category of topological vector spaces. The construction is done by summing appropriate compositions of the initial data over all homotopy classes of successive excisions of embedded pair of pants. We provide sufficient conditions to guarantee these infinite sums converge and as a result, we can generate mapping class group invariant vectors $\Omega_{\mathbf{\Sigma}}$ which we call amplitudes. The initial data encode the amplitude for pair of pants and tori with one boundary, as well as the "recursion kernels" used for glueing. We give this construction the name of "geometric recursion", abbreviated GR.

As an illustration, we show how to apply our formalism to various spaces of continuous functions over Teichmüller spaces, as well as Poisson structures on the moduli space of flat connections. The theory has a wider scope than that and one expects that many functorial objects in low-dimensional geometry and topology should have a GR construction.

The geometric recursion has various projections to topological recursion (TR) and we in particular show it retrieves all previous variants and applications of TR. We also show that, for any initial data for topological recursion, one can construct initial data for GR with values in Frobenius algebra-valued continuous functions on Teichmüller space, such that the $\omega_{g,n}$ of TR are obtained by integration of the GR amplitudes over the moduli space of bordered Riemann surfaces and Laplace transform with respect the boundary lengths. In this regard, the structure of the Mirzakhani-McShane identities served as a prototype of GR.

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1 Introduction

1.1 The geometric recursion

In this article, we develop a machinery to construct *functorial assignments*

$$\mathbf{\Sigma} \mapsto \Omega_{\mathbf{\Sigma}} \in \mathbf{E}(\mathbf{\Sigma}),$$

from a certain category of bordered surfaces to a target $\mathbf{E}$ that needs to be specified together with a small amount of initial data. The target should be a functor from our category of bordered surfaces to a category enriched in vector spaces, together with natural maps handling union and glueing of surfaces. This construction is compatible with natural transformation of targets. We propose to call it "geometric recursion", hereafter GR.

The differential geometry of Teichmüller spaces provides fundamental examples of targets studied in the present article: $\mathbf{E}(\mathbf{\Sigma})$ can be the space of functions and sections of natural bundles on Teichmüller space of $\mathbf{\Sigma}$ or even sections of conformal blocks attached to a given two-dimensional conformal field theory viewed as bundles over Teichmüller spaces. To mention an example of a different kind, we also construct targets from the differential geometry of moduli spaces of flat connections and in fact also geometric objects on such which depends on points in Teichmüller space of the same surface. One can easily imagine a wealth of other geometric structures attached to surfaces which fit in this framework. Other examples of natural candidates for being interesting targets are given by the space of smooth functions, differential forms and cohomology classes, distributions, etc. on Teichmüller spaces but are left for future work.

The functoriality of GR in particular means that $\mathbf{E}(\mathbf{\Sigma})$ carries a representation of the mapping class group $\Gamma(\mathbf{\Sigma})$, and that $\Omega_{\mathbf{\Sigma}}$ is a $\Gamma(\mathbf{\Sigma})$-invariant element of $\mathbf{E}(\mathbf{\Sigma})$, which we call the "GR amplitude". The construction of $\Omega_{\mathbf{\Sigma}}$ is achieved by recursively defining GR amplitudes as a sum over excisions of homotopy classes of embedded pairs of pants from $\mathbf{\Sigma}$. The main difficulty is to show that these infinite sums make sense at each step of the recursion. We provide axioms on the target and the initial data ensuring it is so. Then, functoriality is a direct consequence of the naturality of the construction and the fact that the sum contains all mapping class group translates. The first applications described in Chapter 2 should convince the reader that these axioms are tractable. Our construction is axiomatic, and its effectivity depends on three inputs: (1) the construction of a target in which GR will take values; (2) making the sufficient conditions of admissibility of initial data explicit – *i.e.* ensuring GR makes sense; (3) the explicit computation of the infinite sums, at least for surfaces of low genus/number of boundaries. We explore (1) and (2) in various geometric settings, and we will only briefly touch on (3) for tori with one boundary and four-holed spheres, for now.

We also propose a "strict" version of the geometric recursion, where the source category only knows about the topology of bordered surfaces – and not about mapping classes. The structure of this category is purely combinatorial. It has finitely many objects indexed by (g, n), where g has the meaning of a genus, n of the number of boundary components, and the automorphism group only permutes the boundary components. Objects are glued in a naive way, *e.g.* one can glue $(g_1, 1 + n_1)$ to $(g_2, 1 + n_2)$ to get $(g_1 + g_2, n_1 + n_2)$. Given a strict target theory and a limited amount of initial data, the strict GR amplitudes are defined recursively on $2g - 2 + n$. They are expressed at the end of the induction by finite sums over fatgraphs of genus g with n leaves. Natural transformations from target theories to strict target theories – which therefore forget a lot of information – intertwine GR and strict GR. The reader can keep in mind the following basic example: the target theory is the space of continuous functions on Teichmüller space, the strict target theory is the space of continuous functions on the boundary lengths, and the natural transformation is the integration over a fundamental domain for the mapping class group in the Teichmüller space of surfaces with fixed boundary lengths. As other examples we have in mind, let us mention functions on Teichmüller space valued in a Frobenius algebra, in a modular functor, in a modular operad, etc.

1.2 A geometric interpretation to the topological recursion

Our initial motivation to develop the geometric recursion was the quest for a refinement of the topological recursion of Eynard and Orantin [23], which would incorporate the geometry of surfaces and could be sensitive to geometric structures such as cohomology classes, metrics, connections, etc. Eynard and the third author already observed that the Mirzakhani-McShane identities hinted at the existence of such a formalism, and [25] first attempted to interpret the topological recursion in terms of glueing of surfaces. The identities in question were proved in [39, 40] using hyperbolic geometry and express the constant function 1 on Teichmüller space of a surface Σ as a sum over homotopy classes of embedded pairs of pants in Σ. Integrating this identity over the moduli space implies Mirzakhani's famous recursion on the Weil-Petersson volumes of the moduli space of bordered surfaces, by induction on the Euler characteristic of Σ. Eynard and the third author [24] showed by a Laplace transform computation that the recursion on volumes is equivalent to the topological recursion of [23] (hereafter called TR) for the initial data

$$x(z) = \frac{z^2}{2}, \qquad y(z) = \frac{\sin(2\pi z)}{2\pi}, \qquad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2}. \tag{1}$$

In general, TR is a machinery which produces, for any (g, n) such that $\chi_{g,n} = 2g - 2 + n > 0$, certain generating series of numbers $\omega_{g,n}$ called "correlation functions", by induction on $\chi_{g,n}$. With suitable choices of initial data, TR has many applications in enumerative geometry (topological strings, map enumeration, Hurwitz theory, etc.), knot theory, random matrices, etc. In geometric situations, g and n then receive the respective meaning of the genus and the number of boundaries/punctures of a surface – for instance, for a suitable initial data $\omega_{g,n}$ is a generating series for genus g triangulations with n boundaries. However, from the definition of TR one feels that its geometric meaning is missing. The TR formula has finitely many terms which are in bijection with the mapping class group orbits of the set of homotopy class of embedded pairs of pants in a surface of genus g with n boundaries, the same set which appears in the Mirzakhani-McShane identities. However, in the latter and therefore in TR for Weil-Petersson volumes, each of these terms has a clear meaning, directly provided by the hyperbolic geometry of the embedded pair of pants and the behaviour of geodesics hitting the boundaries. But for the general framework of TR, this correspondence has remained artificial until now. We also remark that TR usually generates numerical invariants (enumeration of surfaces, volumes of moduli spaces, intersection numbers of cohomology classes in cohomological field theories, etc.) only sensitive to boundary information concerning surfaces of genus g with n punctures. The space to which the TR amplitude belong essentially depends on n only.

The strict GR is a generalisation of TR to allow richer glueing maps: the space to which the strict GR amplitudes belong can depend on g and n – like in modular operads. GR itself is much finer, as it is based on a category of bordered surfaces in which automorphism groups are the mapping class groups. We find that any initial data of TR can be lifted to an initial data for the GR corresponding to Frobenius algebra-valued continuous functions over Teichmüller spaces, such that integrating the GR amplitudes over the moduli space retrieves the TR amplitudes after a Laplace transform in the boundary lengths. In the example of (1), this is exactly the mechanism by which the Mirzakhani-McShane identities relate to TR for the Weil-Petersson volumes.

1.3 What should be initial data for the geometric recursion?

Let us introduce and comment on our point of view on initial data. Kontsevich and Soibelman recently reformulated TR (hereafter KS-TR) as a quantisation procedure for a Lagrangian subvariety L defined by quadratic equations in a (possibly infinite-dimensional) symplectic vector space W equipped with a Lagrangian subspace V tangent to L at 0 [37]. The numerical invariants encoded by the TR amplitudes are then collected into a partition function (also called "wave function") Z, defined as a germ of function in V near 0 which is annihilated by a system of at most quadratic differential operators $\hat{L}$ forming a Lie algebra and quantising L. The perturbative resolution of $\hat{L} \cdot Z = 0$ with

$$Z(x) = \exp\Big(\sum_{g,n} \frac{\hbar^{g-1}}{n!} F_{g,n}(x,\ldots,x)\Big), \qquad F_{g,n} \in (V^{\otimes n})^*,$$

directly leads to the structure of the TR formula for $F_{g,n}$. The TR formalism of [23] takes as initial data a spectral curve and it can be embedded in the KS-TR formalism where it corresponds to the case where the Lie algebra is (a direct sum of copies of) the positive part of the Witt algebra. The KS-TR formalism is more general in the sense that the Lie algebra is not bound to be related to the Witt algebra. The initial data of KS-TR is the family of operators $\hat{L}$, whose coefficients are encoded into four tensors (A,B,C,D). The requirement that $\hat{L}$ forms a Lie algebra implies that $\hat{L} \cdot Z = 0$ has non-zero solutions, and it amounts to complete symmetry of A and four quadratic relations involving (A,B,C,D). The role of these tensors in KS-TR is that $A = F_{g=0,n=3}$, $D = F_{g=1,n=1}$ determine the cases $\chi_{g,n} = 1$, while B and C intervene in the recursion relation for $\chi_{g,n} > 1$. The five relations between (A,B,C,D) can be used to prove inductively the complete symmetry of $F_{g,n}$ defined by the KS-TR formula. We noticed in [2] that three of these relations look like coupled IHX-relations, again suggesting the existence of a hidden geometric meaning.

To formulate GR, we find the notion of KS-TR initial data more convenient to adapt than the notion of spectral curve. The initial data for GR therefore consists of functorial elements $A,B,C \in \mathbf{E}(\mathbf{\Sigma}_{0,3})$ and $D \in \mathbf{E}(\mathbf{\Sigma}_{1,1})$. A (respectively D) will represent the amplitude of a pair of pants (respectively, of a torus with one boundary). We *do not* need to impose any algebraic relation between (A,B,C,D) for GR to make sense. The counterpart is that $\Omega_{\mathbf{\Sigma}}$ depends on the choice of a distinguished boundary of $\mathbf{\Sigma}$, a feature which we pass on the choice of our category of surfaces so that morphisms respect the distinguished boundary. On the other hand, it is possible to treat all boundaries in a symmetric way and work with the corresponding category of surfaces. We call this construction "symmetric GR". The initial data (A,B,C,D) for symmetric GR must satisfy an analog of the four quadratic relations appearing in KS-TR. These relations are now geometric in nature. For instance, the H, I, and X terms in each of H-I-X relations represent one mapping class group orbit of the decomposition of a four-holed sphere into two pairs of pants. These four relations allows a recursive proof of functoriality of $\Omega_{\mathbf{\Sigma}}$, most importantly functoriality under braidings involving all boundaries. Although this proof is complicated by homotopy class considerations in the geometry of glueings, its structure (when forming mapping class group orbits) is similar to the proof of symmetry of $F_{g,n}$ in KS-TR [2, Proposition 2.4].

1.4 Main results and outline of the article

Chapter I

Section 2 defines the two main categories of bordered surfaces $\mathrm{Bord}^{\bullet}$ and $\mathrm{Bord}_1^{\bullet}$ we will be concerned with, as well as the glueing and cutting operations, with emphasis on the excision of homotopy class of pairs of pants. The core construction of the geometric recursion appears in Section 3. We introduce the notion of target theory valued in a category $\mathcal{C}$, and the corresponding notion of initial data. Although we choose here a specific category $\mathcal{C}$ built out of topological vector spaces, which is sufficient for most applications involving functional analysis, other choices may be interesting to consider in the future. The data of a target includes a family of "length functions" for curves on surfaces, which are used to tame the countably infinite sums in the definition of GR. Our main result states that the GR formula gives a well-defined functorial assignment (Theorem 3.7), compatible with natural transformations of target theories (Proposition 3.8). The definition and proof that the procedure is well-defined uses an induction on the Euler characteristic. We then proceed to show that the GR amplitudes can be expressed as a fatgraph recursion – where the weight of each fatgraph is given by a sum over all homotopy class of embedding of this fatgraph on the surface.

In Section 4, we further comment on the computation of GR amplitudes for tori with one boundary and for four-holed spheres. In particular, the standard description of the mapping class group cosets in those surfaces here shows that the sums appearing in GR are naturally indexed by paths in the Farey graph. We will not attempt a more explicit description of GR amplitudes in this article.

Section 5 explains a symmetric version of GR, based on the category $\mathrm{Bord}^{\bullet}$ where all boundary components play the same role. Our second main result is that, provided the initial data satisfy four relations involving braiding of boundary components in four-holed spheres and two-holed tori, the same GR formula gives a well-defined functorial assignment from $\mathrm{Bord}^{\bullet}$ (Theorem 5.3).

We observe that 2d topological quantum field theories provide toy examples of strict GR in the first part of Section 7. We also explain that the original topological recursion is a specialisation of strict GR. For pedagogical reasons, we do so in the two languages which have been used to present TR: spectral curves à la Eynard-Orantin and quantum Airy structures à la Kontsevich-Soibelman.

Chapter II

This chapter describes the first examples of target theories, based on various spaces of continuous functions on Teichmüller spaces, and using hyperbolic lengths to specify length functions. Section 8 provides the necessary background in Teichmüller theory. In particular we will use Mirzakhani estimates on the number of multicurves of length $\leq L$, and the geometry of glueing of Teichmüller spaces of pointed bordered surfaces. In order to have a well-defined glueing map, we use marked points on each curve we glue on. The target theories are then constructed in Section 9. We proceed to discuss the integrability of the GR amplitudes with respect to the Weil-Petersson volume form on the moduli space of bordered surfaces (Section 9.5), and the behaviour of the GR formula at the boundary of Teichmüller spaces, *i.e.* when a fixed set of curves are pinched (Section 9.6). We also describe an associated strict target theory, consisting of functions of boundary lengths only, and integration over the moduli space of bordered surfaces yields the natural transformation which intertwines GR and strict GR (Section 9.4).

Mirzakhani-McShane identities provide a fundamental and non-trivial example of initial data, for which the GR amplitude can be explicitly computed: $\Omega_{\boldsymbol{\Sigma}} = 1$ (Section 10.1). Although this initial data is not admissible according to our first definition, the knowledge of Mirzakhani-McShane identities gives us another condition, which we call M-admissibility, under which we show that GR is well-defined and actually produces uniformly bounded functions on the Teichmüller space (Section 10.2). If we did not know Mirzakhani-McShane identities, the symmetry of the GR amplitude 1 under braidings of all boundary components would be mysterious. It would be explained inductively *e.g.* if Mirzakhani initial data satisfied the symmetry axioms, which here boil down to two pointwise relations in the Teichmüller space of a sphere with four boundary components between certain sums over simple closed curves. We know these relations hold after integration over the moduli space, but we do not know whether they are true pointwise (Section 10.4).

The strict GR amplitudes which are functions of the boundary lengths L can be alternatively be described *via* their Laplace transform, as functions of variables $(z_1, \ldots, z_n)$. We examine the Laplace transform of the strict GR formula in Sections 11 and 12. Multiplying lengths in Mirzakhani initial data by a parameter $\beta \geq 1$ and sending $\beta \to \infty$ gives an M-admissible initial data for GR. The integration of its GR amplitudes over the moduli space of bordered surfaces of fixed boundary lengths becomes a polynomial generating series for the intersection numbers of ψ-classes on $\overline{\mathcal{M}}_{g,n}$ – *aka* Witten-Kontsevich model – computed by the corresponding strict GR (Section 12.1). By taking its Laplace transform of the strict GR formula, we retrieve the residue formula of Eynard-Orantin TR for the spectral curve $x = y^2$ (Section 12.2).

This leads us to describe in Theorem 12.6-12.8 and Section 12.6 a Laplace inverse transform description of TR for any spectral curve with simple ramification points, *i.e.* in terms of boundary lengths L instead of points on the spectral curve z. This applies in particular to the correlation functions of semi-simple cohomological field theories. The answer involves distributions – the Dirac and its derivatives. Regularising with the parameter β allows lifting any initial data for TR to an initial data for GR, such that the integration over the moduli space of its GR amplitudes become after Laplace transform and in the limit $\beta \to \infty$ the TR amplitudes. The GR amplitudes themselves may become distributions when $\beta \to \infty$, and it is a pleasant specificity of the Witten-Kontsevich model that its $\beta = \infty$ GR amplitudes exist as continuous functions.

We describe in Section 10.3 an important idea of "fibering" a target theory over Teichmüller space, *i.e.* looking at continuous functions over Teichmüller space valued in some other target theory. This is illustrated in Section 13 where we define a target theory out of sections of a modular functor with vanishing central charge and conformal dimensions – *e.g.* the endomorphism theory of a modular functor.

In Section 14, we sketch a target theory which can be used to construct geometric structures on the moduli space of flat connections, using ideas from lattice gauge theory and especially the approach of Fock-Rosly. We apply it to obtain the Fock-Rosly Poisson structure [30] from GR.

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Part I

First principles of the geometric recursion

2 Two-dimensional geometry background

2.1 Bordered surfaces

Definition 2.1 *A bordered surface Σ is a smooth, oriented, compact 2-dimensional manifold with an orientation of its boundary.*

We remark that a bordered surface has a decomposition of its boundary, namely the subset where the induced orientation agrees with the given orientation and the subset where they disagree. We denote these two subsets of the boundary of Σ, $\partial_+\Sigma$ and $\partial_-\Sigma$ respectively. We have of course that $\partial_\pm\Sigma$ consist of a subset of the connected components of $\partial\Sigma$ and that

$$\partial\Sigma = \partial_+\Sigma \cup \partial_-\Sigma.$$

We use the notation Σ° for the interior of Σ. If $b \in \pi_0(\partial\Sigma)$, we denote the corresponding oriented component $\partial_b\Sigma$.

Definition 2.2 *A bordered surface Σ is stable if it is empty, or if each component has a non-empty boundary and has negative Euler characteristic. It is unstable otherwise.*

The mapping class group Γ_Σ consists of the set of isotopy classes of orientation preserving diffeomorphisms, which also preserve the prescribed orientation on the boundary.

We use the following notation for the components of Σ

$$\Sigma = \bigcup_{a\in\pi_0(\Sigma)} \Sigma(a).$$

2.2 Pointed bordered surfaces

Definition 2.3 *A pointed bordered surface $\mathbf{\Sigma}$ is a bordered surface Σ together with a marked point $o_b \in \partial_b\Sigma$ for each $b \in \pi_0(\partial\Sigma)$.*

We consider the category of pointed bordered surfaces $\mathrm{Bord}^\bullet$. Its objects are pointed bordered surfaces, and its morphisms are isotopy classes of orientation preserving diffeomorphisms $f : \Sigma \to \Sigma'$ sending $\partial_\pm\Sigma$ and its set of distinguished points to $\partial_\pm\Sigma'$, and its set of distinguished points. We observe that $\mathrm{Bord}^\bullet$ is a symmetric monoidal category for the disjoint union, with unit the empty surface.

If $\mathbf{\Sigma}$ is a pointed bordered surface, we denote by $\mathrm{Aut}(\mathbf{\Sigma}) = \Gamma_{\mathbf{\Sigma}}$ the corresponding mapping class group. It is the set of isotopy classes of orientation preserving diffeomorphisms which preserves the set of marked points on the boundary. It is isomorphic to the central extension of Γ_Σ by isotopy classes of Dehn twists along curves in Σ° homotopic to the boundary components.

Let $\Delta_{\mathbf{\Sigma}}$ be the group generated by the boundary parallel Dehn twists. We then have an exact sequence

$$1 \longrightarrow \Delta_{\mathbf{\Sigma}} \longrightarrow \Gamma_{\mathbf{\Sigma}} \longrightarrow \Gamma_\Sigma \longrightarrow 1. \tag{2}$$

For any homotopy class of a simple closed curve β in Σ°, we denote by δ_β the corresponding Dehn twist.

2.3 Marked pointed bordered surfaces

We will further need to consider marked pointed bordered surfaces, e.g. pointed bordered surfaces with marked points in the interior.

Definition 2.4 *A marked pointed bordered surface $\mathbf{\Sigma}$, is a pointed bordered surface Σ together with a finite set of marked points $P \subset \Sigma^\circ$. For a pointed bordered surface $\mathbf{\Sigma} = (\Sigma, o)$, we define its* double *as the following marked surface*

$$\mathbf{\Sigma}^d = (\Sigma \cup_\partial \overline{\Sigma}, o).$$

where $\overline{\Sigma}$ is the surface with opposite orientation.

We observe that the double $\mathbf{\Sigma}^d$ of $\mathbf{\Sigma}$ is a closed surface and that it has an orientation-reversing involution $r : \mathbf{\Sigma}^d \to \mathbf{\Sigma}^d$ induced from the identity map of $\mathbf{\Sigma}$, but which interchanges the two copies of $\mathbf{\Sigma}$ in $\mathbf{\Sigma}^d$.

If we let $\Gamma^{\partial\mathbf{\Sigma}}_{\mathbf{\Sigma}^d}$ be the stabiliser of the isotopy class of each component of $\partial\Sigma \subset \Sigma^d$ in $\Gamma_{\mathbf{\Sigma}^d}$, then we have the short exact sequence

$$1 \to \Gamma_{\mathbf{\Sigma}} \to \Gamma^{\partial\mathbf{\Sigma}}_{\mathbf{\Sigma}^d} \to (\mathbb{Z}/2\mathbb{Z})^{\pi_0(\Sigma)} \to 1.$$

We observe that for any marked pointed bordered surface $\mathbf{\Sigma}$, we have the underlying pointed bordered Σ.

Notation 2.5 *If $\mathbf{\Sigma}$ is a (marked) pointed bordered surface, the underlying surface without its marked points is denoted Σ. We keep bold letters whenever the structure given by the marked points matters, and use normal letters otherwise.*

2.4 Glueing and cutting operations

Let $\mathbf{\Sigma}$ be a stable pointed bordered surface, and select a $b = (b', b'') \in C_2(\pi_0(\partial\Sigma))$, where $C_2(\pi_0(\partial\Sigma))$ is the set of ordered distinct pairs of elements from $\pi_0(\partial\Sigma)$. Choose arbitrarily an orientation-reversing (with respect to the induced orientations from Σ) diffeomorphism $\varphi : (\partial_{b'}\Sigma, o_{b'}) \to (\partial_{b''}\Sigma, o_{b''})$. We construct another pointed bordered surface $\mathbf{\Sigma}_{b,\varphi} = \mathbf{\Sigma}/\sim$, where $\sim$ is the equivalence relation generated by φ. The projection gives a continuous map $p : \mathbf{\Sigma} \to \mathbf{\Sigma}_{b,\varphi}$.

Lemma 2.6 *For any two glueing maps φ and φ', $\mathbf{\Sigma}_{b,\varphi}$ is diffeomorphic to $\mathbf{\Sigma}_{b,\varphi'}$, canonically so up to isotopy.*

Proof. The map $\varphi' \circ \varphi^{-1} : (\partial_{b'}\Sigma, o_{b'}) \to (\partial_{b'}\Sigma, o_{b'})$ is isotopic to the identity. Choose an extension $f : \Sigma \to \Sigma$ of it, which is the identity outside of a small annular neighbourhood of $\partial_{b'}\Sigma$, and which is isotopic to the identity. By construction, the induced continuous map $p' \circ f : \mathbf{\Sigma} \to \mathbf{\Sigma}_{b,\varphi'}$ descends to a continuous map $\tilde{f} : \mathbf{\Sigma}_{b,\varphi} \to \mathbf{\Sigma}_{b,\varphi'}$ such that $\tilde{f} \circ p = p' \circ f$. We can always choose f such that $\tilde{f}$ is a diffeomorphism. Moreover, any two such choices of f yield isotopic $\tilde{f}$'s. ■

Since we have this canonical identification of $\mathbf{\Sigma}_{b,\varphi}$ with $\mathbf{\Sigma}_{b,\varphi'}$, we simply leave out the glueing map from the notation and just denote the glued surface $\mathbf{\Sigma}_b$. We observe that if we have a finite set b of pairwise disjoint elements of $C_2(\pi_0(\partial\Sigma))$, then we can also define $\mathbf{\Sigma}_b$ in a similar way by glueing along all the pairs in b.

Conversely, we have the operation of cutting a stable pointed bordered surface $\mathbf{\Sigma}$ on a pointed oriented simple closed curve (γ, o) (for short just γ) in Σ°, so as to obtain a new pointed bordered $\mathbf{\Sigma}^\gamma$, where we induce the orientation of the curve on the new boundary component (denoted $\partial^+_\gamma\Sigma^\gamma$) to the left of γ and the opposite orientation on the new component on the right (denoted $\partial^-_\gamma\Sigma^\gamma$). This means that all the new boundary components of Σ^γ are in $\partial_+\Sigma^\gamma$. We further remark, that if we have an orientation-preserving homotopy from γ to some other oriented simple closed γ' (we allow moving the distinguished point), we obtain diffeomorphisms from $\mathbf{\Sigma}^\gamma$ to $\mathbf{\Sigma}^{\gamma'}$, whose isotopy class is uniquely determined by the homotopy.

Definition 2.7 *The homotopy class of a one-dimensional compact submanifold in Σ is called a multicurve. The set of multicurves in Σ is denoted S_Σ. The set of multicurves which are not boundary-parallel is denoted S°_Σ. We denote $\hat{S}^\circ_\Sigma$ the subset of simple closed curves, i.e. multicurves with a single connected component.*

Definition 2.8 *A pointed (oriented) multicurve (γ, o) (for short just γ) on a pointed bordered surface $\mathbf{\Sigma}$ is the homotopy class of a finite disjoint union of one-dimensional pointed, compact (oriented), connected submanifold in Σ. They form a set $S_{\mathbf{\Sigma}}$. Adding the assumption that none of the components are boundary-parallel gives a subset $S^\circ_{\mathbf{\Sigma}}$.*

We define the cutting and its notation for pointed oriented multicurves in complete analogy with the cutting for a simple pointed oriented curve.

Definition 2.9 *A multicurve is called stable, if Σ^γ is stable.*

If $\mathbf{\Sigma}^\gamma$ is obtained from $\mathbf{\Sigma}$ by cutting along a stable pointed oriented multicurve γ in Σ, we have the exact sequence of groups

$$1 \longrightarrow \Delta_\gamma \longrightarrow \Gamma_{\mathbf{\Sigma}^\gamma} \longrightarrow \mathrm{Stab}(\gamma) \longrightarrow 1, \qquad \Delta_\gamma := \prod_{\beta \in \pi_0(\gamma)} \langle \delta_{\beta^+} \delta_{\beta^-}^{-1}, \rangle \tag{3}$$

where $\mathrm{Stab}(\gamma) \subseteq \Gamma_{\mathbf{\Sigma}^\gamma}$ are the mapping classes which preserve the image of γ in Σ^γ, and β^+ and β^- are the preimages of β in Σ^γ.

Definition 2.10 *A* self-glueing *is a glueing in which at least two boundary components of the same connected surface are identified.*

2.5 Excising a pair of pants

The case which is particularly relevant for us is the excision of a pair of pants (*i.e.* a genus 0 surface with 3 boundary components) from a stable pointed bordered surface Σ of genus g with n boundaries. We assume $2g - 2 + n > 1$.

Let $f : \mathbf{P} \to \mathbf{\Sigma}$ be an embedding of a pointed pair of pants. We require that when a boundary component b of P is mapped to a boundary component b' of Σ, the marked point in b should be mapped to the marked point in b', but we allow some of the boundary components of P to be mapped to Σ°. We further require that $f(P)$ contains at least one of the boundary components of Σ. Let us cut Σ along the boundary components of $f(P)$, which are not boundaries of Σ. The assumption $2g - 2 + n > 1$ implies that $\Sigma' = \Sigma \setminus f(P^\circ)$ is stable. We choose the induced orientation from Σ' on the boundary components of P which belong to Σ°. The marked points in the boundary components of P give the structure of a pointed bordered surface on Σ'. Later, in certain cases, we will change this orientation. There are finitely many possible topologies for Σ' and $P \subset \Sigma$.

- **I** $(g, n) \neq (1, 1)$, $g \geq 1$, P has exactly one boundary component common with Σ and Σ' is connected (of genus $g - 1$ with $n + 1$ boundaries).
- **I'** $(g, n) \neq (0, 3)$, P has exactly one boundary component γ disjoint from $\partial\Sigma$ (we denote $\{b, b'\}$ the two others) and Σ' is connected of genus g with $n - 1$ boundaries.
- **II** P has exactly one boundary component b common with Σ (we denote $\{\gamma, \gamma'\}$ the two others), and Σ' has two connected components with genus $\{g_1, g_2\}$ and number of boundaries $\{n_1, n_2\}$ such that $g_1 + g_2 = g$, $n_1 + n_2 = n + 1$ and $(g_i, n_i) \neq (0, 1), (0, 2)$ for any $i \in \{1, 2\}$.

We now describe the precise way we will enumerate embedded pairs of pants. >From now on we will assume that an ordering on the boundary components of $\mathbf{P}$ has been chosen, *i.e.* a bijection between $\{1,2,3\}$ and $\pi_0(\partial P)$.

For $b_1, b \in \pi_0(\partial\Sigma)^{\times 2}$, we denote $\mathfrak{C}_{b_1,b}(\mathbf{\Sigma})$ the set of homotopy classes of simple curves in Σ with endpoints the marked points in $\partial_{b_1}\Sigma \cup \partial_b\Sigma$ modulo the group generated by the Dehn twists δ_{b_1} and δ_b with the following condition in the case that $b_1 = b$. In this case we require that $c \in \mathfrak{C}_{b,b}(\mathbf{\Sigma})$ is not homotopic to any composition of the form $\tilde{c}^{-1} \cdot (\partial_{b'}\Sigma) \cdot \tilde{c}$ for a $\tilde{c} \in \mathfrak{C}_{b,b'}(\mathbf{\Sigma})$, for some $b' \neq b$, where we use $\cdot$ to denote the composition of curves in the path groupoid of Σ.

Suppose first that $b_1 \neq b$ (case **I'**). Then for any $c \in \mathfrak{C}_{b_1,b}(\mathbf{\Sigma})$, there exists an embedding $f : \mathbf{P} \to \mathbf{\Sigma}$ such that $f(P)$ contains a representative from c with $f(\partial_1 P) = \partial_{b_1}\Sigma$ and $f(\partial_2 P) = \partial_b\Sigma$. Any two such embeddings of P are related by the action of the group of diffeomorphisms of Σ which preserve the marked points on the boundary of Σ, and which are isotopic to the identity among such. This means that for any two embeddings $f_i : \mathbf{P} \to \mathbf{\Sigma}$ as before, there exists a morphism of pointed bordered surfaces $\Phi : \mathbf{\Sigma}^{f_1(\partial_3 P)} \to \mathbf{\Sigma}^{f_2(\partial_3 P)}$ induced by the above group action, which is unique up to the action of $\Delta_{f_2(\partial_3 P)}$. We observe therefore that the embedding of the pair of pants modulo this natural equivalence is determined by the isotopy class of the pointed oriented curve $\gamma_c := f(\partial_3 P)$ modulo the action of Δ_{γ_c}. We define the two pointed bordered surfaces $\mathbf{P}_c$ and $\mathbf{\Sigma}_c$ such that $\mathbf{\Sigma}^{\gamma_c} = \mathbf{P}_c \cup \mathbf{\Sigma}_c$. We note that this notation should not be confused with the notation for the glueing of surface along a number of pairs of boundary components. This should however not be a problem, since c here is a completely different object.

In the case where $b_1 = b$ (either **I** or **II**), we get that for any $c \in \mathfrak{C}_{b_1,b_1}(\mathbf{\Sigma})$ there exists an embedding $f : \mathbf{P} \to \mathbf{\Sigma}$ such that $f(P)$ contains a representative from c, with $f(\partial_1 P) = \partial_{b_1}\Sigma$ and such that cutting P along the inverse image under f of the representative of c puts $\partial_2 P$ in a different component than $\partial_3 P$. Any two such embeddings of P are related by the action of diffeomorphisms of Σ which preserve the marked points on the boundary of Σ, and which are isotopic to the identity among such. This means that for any two such embeddings $f_i : \mathbf{P} \to \mathbf{\Sigma}$, there exists a morphism of pointed bordered surfaces $\Phi : \mathbf{\Sigma}^{f_1(\partial_2 P \cup \partial_3 P)} \to \mathbf{\Sigma}^{f_2(\partial_2 P \cup \partial_3 P)}$ induced by the above group actions, which is unique up to the action of $\Delta_{f_2(\partial_2 P \cup \partial_3 P)}$. We observe thus that the embedding of the pair of pants modulo this natural equivalence is determined by the isotopy class of the pointed oriented multicurve $\gamma_c := f_2(\partial_2 P \cup \partial_3 P)$ together with the choice of a first component – here this is $\gamma_c^1 := f_2(\partial_2 P)$, and the second component is denoted $\gamma_c^2 := f_2(\partial_3 P)$, all of this modulo the action of Δ_{γ_c}. It is assumed here that the isotopies preserve the choice of the first component. We define the two pointed bordered surfaces $\mathbf{P}_c$ and $\mathbf{\Sigma}_c$ such that $\mathbf{\Sigma}^{\gamma_c} = \mathbf{P}_c \cup \mathbf{\Sigma}_c$ and by changing the orientation of the boundary component of $\mathbf{\Sigma}$ associated with γ_c^1, so as to force it to disagree with the orientation of $\mathbf{\Sigma}_c$. The reason for this change of orientation will become clear in the next section.

For a fixed $b_1 \in \pi_0(\partial\Sigma)$, the mapping class group acts on

$$\mathfrak{C}_{b_1}(\mathbf{\Sigma}) := \bigcup_{b \in \pi_0(\partial\Sigma)} \mathfrak{C}_{b_1,b}(\mathbf{\Sigma})\,.$$

This action has finitely many orbits, which have been classified in **I**, **I'** and **II**. In particular, there is a unique orbit for each $b \neq b_1$, which we denote $\mathcal{O}^b$.

2.6 The categories $\mathrm{Bord}_1^\bullet$ and $\mathrm{Bord}_s^\bullet$

We consider the full subcategory $\mathrm{Bord}_1^\bullet$ of $\mathrm{Bord}^\bullet$, where we require that objects $\mathbf{\Sigma}$ are stable and such that the natural map from $\pi_0(\partial_-\Sigma)$ to $\pi_0(\Sigma)$ is a bijection. For disconnected objects $\mathbf{\Sigma}$ of $\mathrm{Bord}_1^\bullet$, we denote similarly the connected components by $(\mathbf{\Sigma}(a))_{a\in\pi_0(\Sigma)}$.

We observe that the above discussion about cutting and glueing in $\mathrm{Bord}^\bullet$ applies to $\mathrm{Bord}_1^\bullet$ with the following important modifications. If we have an object $\mathbf{\Sigma}$ in $\mathrm{Bord}_1^\bullet$ and γ is a stable pointed oriented multicurve in $\mathbf{\Sigma}$, we can construct $\mathbf{\Sigma}^\gamma$ as an object in $\mathrm{Bord}^\bullet$ as described above. However, it is not clear that this object belongs to $\mathrm{Bord}_1^\bullet$.

Definition 2.11 *A* headed *multicurve in an object* $\mathbf{\Sigma}$ *or* $\mathrm{Bord}_1^\bullet$ *is an ordered pair* (γ,β) *where* γ *is a multicurve in* Σ *and* β *is a subset of* $\pi_0(\partial_+\Sigma^\gamma)$ *such that the natural inclusion of* $\pi_0(\partial_-\Sigma^\gamma)\cup\beta$ *into* $\pi_0(\Sigma^\gamma)$ *is a bijection.*

If (γ,β) is a stable headed pointed oriented multicurve in $\mathbf{\Sigma}$, we define $\mathbf{\Sigma}^{\gamma,\beta}$ to be the object in $\mathrm{Bord}_1^\bullet$ which results from reversing the orientation of those boundary components of $\mathbf{\Sigma}^\gamma$ which are in β.

Definition 2.12 *If* $\mathbf{\Sigma}$ *is a connected object in* $\mathrm{Bord}_1^\bullet$*, we will simply denote* $\mathfrak{C}_b(\mathbf{\Sigma})$ *by* $\mathfrak{C}(\mathbf{\Sigma})$*, where* $b = \pi_0(\partial_-\Sigma)$*.*

We remark that excising a pair of pants specified by $c\in\mathfrak{C}(\mathbf{\Sigma})$ with the definitions of Section 2.5 canonically specifies a stable headed pointed oriented multicurve γ_c and an object $\mathbf{\Sigma}_c := \mathbf{\Sigma} - \mathbf{P}_c$ in $\mathrm{Bord}_1^\bullet$.

Conversely, if we consider glueing, even starting with an object $\mathbf{\Sigma}$ in $\mathrm{Bord}_1^\bullet$ and an ordered pair $b\in C_2(\pi_0(\partial\Sigma))$, then $\mathbf{\Sigma}_b$ may not be in $\mathrm{Bord}_1^\bullet$ again. By convention, the only glueing operations allowed on an object $\mathbf{\Sigma}$ in $\mathrm{Bord}_1^\bullet$, are those glueing operations on $\mathbf{\Sigma}$ considered as an object in $\mathrm{Bord}^\bullet$, that produce an object $\mathbf{\Sigma}_b$ in $\mathrm{Bord}_1^\bullet$.

We will also consider the full subcategory $\mathrm{Bord}_\mathrm{s}^\bullet$ of $\mathrm{Bord}^\bullet$, where we require that objects $\mathbf{\Sigma}$ are stable, and if $\Sigma\neq\varnothing$ it must have a non-empty boundary and $\partial_-\Sigma=\varnothing$. In other words, all boundary components play a symmetric role. All glueing/cutting operations from $\mathrm{Bord}^\bullet$ are allowed in $\mathrm{Bord}_\mathrm{s}^\bullet$, provided at the end, we make the choice to orient all boundary components in agreement with the surface itself.

When the category has been specified, we shall denote $\Gamma(\mathbf{\Sigma})$ the automorphism group of $\mathbf{\Sigma}$, and call it indifferently the mapping class group. We also denote $\Gamma^\partial(\mathbf{\Sigma})$ the pure mapping class groups, *i.e.* mapping classes inducing as permutation of $\pi_0(\partial\Sigma)$ the identity map. The mapping class groups for pointed bordered surfaces and for bordered surfaces will often occur and for them we use the special notation $\Gamma_{\mathbf{\Sigma}}$ and Γ_Σ as described in Sections 2.1-2.3.

3 Geometric recursion

Our goal is to construct, by means of sums over all pair of pants decompositions, for any stable $\mathbf{\Sigma}$ in $\mathrm{Bord}_1^\bullet$, an element $\Omega_{\mathbf{\Sigma}}$ of a "space" $\mathbf{E}(\mathbf{\Sigma})$ attached to the surface $\mathbf{\Sigma}$, which is a functorial assignment. First of all we need to specify what is the home $\mathbf{E}(\mathbf{\Sigma})$ for $\Omega_{\mathbf{\Sigma}}$, and for the construction to be meaningful, it should incorporate a compatible set of union and glueing maps, and a mechanism which can make sense of the "countable sums over pair of pants decompositions". We call this a *target theory.* To be precise, it will be a functor $\mathbf{E}$ from $\mathrm{Bord}_1^\bullet$ to some category $\mathcal{C}$, satisfying a list of axioms to be specified in Section 3.1. The definition of $\Omega_{\mathbf{\Sigma}}$ will depend on a small amount of *initial data*, specifying what happens for a pair of pants, for a torus with one boundary, and a way to inductively increase the complexity of the surfaces (Section 3.2). We call the inductive process leading to $\Omega_{\mathbf{\Sigma}}$ from such initial data *geometric recursion* (in the chosen target theory $\mathbf{E}$), abbreviated GR.

We also give a variant of this construction based on $\mathrm{Bord}_\mathrm{s}^\bullet$, which we call *symmetric GR.* Although the defining formulae are essentially the same, we have to identify consistency conditions – spelled out in Section 5.1 – that the initial data must satisfy to guarantee that the assignment $\mathbf{\Sigma} \mapsto \Omega_{\mathbf{\Sigma}}$ is functorial. The key point here is that, unlike GR, symmetric GR includes invariance under mapping classes which can permute all boundary components.

3.1 Target theories

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. We choose $\mathcal{C}$ to be constructed from the category $\mathcal{V}$ of Hausdorff, complete, locally convex topological vector spaces over $\mathbb{K}$ as described below. We remark that $\mathcal{V}$ admits projective limits [48].

Recall that an object V of $\mathcal{V}$ comes with a (possibly uncountable) family of seminorms $(|\cdot|_\alpha)_{\alpha\in\mathscr{A}}$ which induces the topology of the vector space V such that it is Hausdorff and complete as a locally convex topological vector space.

Definition 3.1 *An object* $\mathbf{V}$ *of* $\mathcal{C}$ *is a directed set* $\mathscr{I}$ *and an inverse system over* $\mathscr{I}$ *of objects*

$$(V^{(i)}, (|\cdot|_\alpha)_{\alpha\in\mathscr{A}^{(i)}})_{i\in\mathscr{I}}$$

of $\mathcal{V}$*, together with an object* V *of* $\mathcal{V}$ *which is the projective limit of the* $(V^{(i)})_{i\in\mathscr{I}}$*. We then define the (possibly infinite) seminorms* $\|\cdot\|_i$ *by*

$$\|v\|_i := \sup_{\alpha\in\mathscr{A}^{(i)}} |v|_\alpha \in [0, +\infty]$$

and the (possibly not closed) subspace

$$V' := \{v \in V \quad | \quad \forall i \in \mathscr{I},\ \|v\|_i < +\infty\} \subset V\,. \tag{4}$$

A morphism $\mathbf{\Phi}$ *of* $\mathcal{C}$ *from an object* $\mathbf{V}_1$ *to* $\mathbf{V}_2$*, is an inverse system of continuous linear maps*

$$\Phi^{i,j} : V_1^{(i)} \to V_2^{(j)}, \qquad i \in \mathscr{I}_1, \quad j \le h(i)$$

over an order preserving map $h : \mathscr{I}_1 \to \mathscr{I}_2$*, such that the induced continuous linear map* $\Phi : V \to W$ *satisfies* $\Phi(V_1') \subseteq V_2'$*.*

This category is suited to treat in a uniform way spaces of functions, forms, distributions, sections of vector bundles on many different topological spaces with various structures associated to surfaces, such as various moduli spaces of different kinds of structures on surfaces, possibly coupled to Teichmüller space, etc. It seems to contain a rather minimal structure to make sense of GR. Applications may justify other choices of categories. For instance, the category of $\mathbb{Z}$-graded complexes of objects in the above $\mathcal{C}$.

Definition 3.2 *Given three objects* $\mathbf{V}_j$*,* $j = 1, 2, 3$ *of* $\mathcal{C}$*, a bilinear morphism*

$$\mathbf{F} : \mathbf{V}_1 \times \mathbf{V}_2 \to \mathbf{V}_3,$$

is an inverse system of continuous bilinear maps

$$F^{i,j,k} : V_1^{(i)} \times V_2^{(j)} \to V_3^{(k)} \qquad (i,j) \in \mathscr{I}_1 \times \mathscr{I}_2, \quad k \le h(i,j)$$

over an order preserving map $h : \mathscr{I}_1 \times \mathscr{I}_2 \to \mathscr{I}_3$ *(where the lexicographic order governs the source), such that the induced bilinear map* $F : V_1 \times V_2 \to V_3$ *satisfies* $F(V_1' \times V_2') \subseteq V_3'$*. Multlinear morphisms are defined in a similar fashion.*

We do not consider tensor products of topological vector spaces, which usually turn bilinear maps into linear ones, since we want to (and we can) avoid discussion regarding completions. This allow more theories to naturally fit in the general setup.

Definition 3.3 *A ($\mathcal{C}$-valued) target theory is a functor* $\mathbf{E}$ *from* $\mathrm{Bord}_1^\bullet$ *to the category* $\mathcal{C}$ *together with the following extra structure. For each object* $\mathbf{\Sigma}$ *of* $\mathrm{Bord}_1^\bullet$ *with*

$$\mathbf{E}(\mathbf{\Sigma}) = \big(E^{(i)}(\mathbf{\Sigma}), (|\cdot|_\alpha)_{\alpha\in\mathscr{A}_{\mathbf{\Sigma}}^{(i)}}\big)_{i\in\mathscr{I}_{\mathbf{\Sigma}}},$$

we require the functorial data of lengths functions

$$l_\alpha : S_\Sigma \longrightarrow \mathbb{K} \smallsetminus \{0\}$$

indexed by $i \in \mathscr{I}_{\mathbf{\Sigma}}$ *and* $\alpha \in \mathscr{A}_{\mathbf{\Sigma}}^{(i)}$. *This data must satisfy the following properties.*

Polynomial growth axiom. *For each* $i \in \mathscr{I}_{\mathbf{\Sigma}}$, $\alpha \in \mathscr{A}_{\mathbf{\Sigma}}^{(i)}$ *and* $L \in \mathbb{R}_{>0}$, *the set*

$$N_\alpha(\mathbf{\Sigma}, L) = \big\{\gamma \in S_\Sigma^\circ \quad \big| \quad |l_\alpha(\gamma)| \le L\big\}$$

is finite and there exists $m_i, d_i \in \mathbb{R}_{>0}$, *such that we have an upper bound on the cardinality*

$$\sup_{\alpha\in\mathscr{A}^{(k)}(\mathbf{\Sigma})} |N_\alpha(\mathbf{\Sigma}, L)| \le m_i L^{d_i}.$$

Lower bound axiom. *For any* $i \in \mathscr{I}_{\mathbf{\Sigma}}$, *there exists* $\varepsilon_i > 0$ *such that*

$$\inf\big\{|l_\alpha(\gamma)| \quad \big| \quad (\alpha, \gamma) \in \mathscr{A}_{\mathbf{\Sigma}}^{(i)} \times S_\Sigma^\circ\big\} \ge \varepsilon_i.$$

Union axiom. *For any two objects* $\mathbf{\Sigma}_1$ *and* $\mathbf{\Sigma}_2$ *of* $\mathrm{Bord}_1^\bullet$, *we ask for a bilinear morphism*

$$\sqcup : \mathbf{E}(\mathbf{\Sigma}_1) \times \mathbf{E}(\mathbf{\Sigma}_2) \to \mathbf{E}(\mathbf{\Sigma}_1 \cup \mathbf{\Sigma}_2),$$

compatible with associativity of cartesian products and associativity of unions. We require $E(\varnothing) := \mathbb{K}$, *and the union map* $\sqcup : \mathbf{E}(\varnothing) \times \mathbf{E}(\mathbf{\Sigma}) \to \mathbf{E}(\mathbf{\Sigma})$ *is specified by* $1 \sqcup v = v$.

Glueing axiom. *For any two objects* $\mathbf{\Sigma}_1$ *and* $\mathbf{\Sigma}_2$ *in* $\mathrm{Bord}_1^\bullet$, *and a subset* $b \subset \pi_0(\partial\Sigma_1) \times \pi_0(\partial\Sigma_2)$ *consisting of disjoint pairs, let* $\mathbf{\Sigma}$ *be the object of* $\mathrm{Bord}_1^\bullet$ *obtained from glueing* $\mathbf{\Sigma}_1$ *to* $\mathbf{\Sigma}_2$ *along* b. *We ask for the data of a bilinear morphism*

$$\mathbf{\Theta}_b : \mathbf{E}(\mathbf{\Sigma}_1) \times \mathbf{E}(\mathbf{\Sigma}_2) \to \mathbf{E}(\mathbf{\Sigma}),$$

which is compatible with the glueing of morphisms, with associativity of glueings and with the union morphisms.

One can formulate likewise the notion of a symmetric target theory by replacing $\mathrm{Bord}_1^\bullet$ with its symmetric analog $\mathrm{Bord}_{\mathrm{s}}^\bullet$ – see Section 2.

Cutting along a stable headed oriented pointed multicurve (γ, β) a connected object $\mathbf{\Sigma}$ of $\mathrm{Bord}_1^\bullet$ and performing the reciprocal glueing operation gives rise to a glueing morphism

$$\mathbf{\Theta}_{\gamma,\beta} : \prod_{a\in\pi_0(\Sigma^{\gamma,\beta})} \mathbf{E}(\mathbf{\Sigma}^{\gamma,\beta}(a)) \to \mathbf{E}(\mathbf{\Sigma}).$$

If we have two stable headed oriented pointed multicurves (γ,β) and (γ',β') in the same $\Gamma(\mathbf{\Sigma})$-orbit $\mathcal{O}$, we can pick a morphism $\Phi : \mathbf{\Sigma}^{\gamma,\beta} \to \mathbf{\Sigma}^{\gamma',\beta'}$ in $\mathrm{Bord}_1^{\bullet}$, which induces an isomorphism of the mapping class group invariant subspaces

$$\left(\prod_{a\in\pi_0(\Sigma^{\gamma,\beta})} \mathbf{E}(\mathbf{\Sigma}^{\gamma,\beta}(a))\right)^{\Gamma(\mathbf{\Sigma}^{\gamma,\beta})} \cong \left(\prod_{a'\in\pi_0(\Sigma^{\gamma',\beta'})} \mathbf{E}(\mathbf{\Sigma}^{\gamma',\beta'}(a'))\right)^{\Gamma(\mathbf{\Sigma}^{\gamma',\beta'})}.$$

By construction this isomorphism is independent of the choice of Φ. So, all these spaces can thus be identified to a single cartesian product $\mathbf{E}_{\mathcal{O}}$ by canonical identifications, and the glueing morphisms induce well-defined continuous multlinear morphisms $\mathbf{\Theta}_{\gamma,\mathcal{O}} : \mathbf{E}_{\mathcal{O}} \to \mathbf{E}(\mathbf{\Sigma})$. We shall often resort to this argument allowing the identification of the source of all glueing maps within the same mapping class group orbit.

In particular, if we have a $c \in \mathfrak{C}_{b_1}(\mathbf{\Sigma})$, we can excise the embedded pair of pants $\mathbf{P}_c$ as explained in Section 2.5. We denote the resulting morphism

$$\mathbf{\Theta}_c : \mathbf{E}(\mathbf{P}_c) \times \mathbf{E}(\mathbf{\Sigma}_c) \to \mathbf{E}(\mathbf{\Sigma})$$

with its associated order preserving map

$$h_c : \mathscr{I}_{\mathbf{P}_c} \times \mathscr{I}_{\mathbf{\Sigma}_c} \to \mathscr{I}_{\mathbf{\Sigma}},$$

which are induced by the reglueing of $\mathbf{P}_c$ and $\mathbf{\Sigma}_c$ naturally identified with $\mathbf{\Sigma}$.

3.2 Initial data

All surfaces in the remaining of Section 3 are objects in $\mathrm{Bord}_1^{\bullet}$.

Definition 3.4 Initial data *for a given target theory* $\mathbf{E}$ *are functorial assignments*

- *of* $A_{\mathbf{P}}, C_{\mathbf{P}} \in \mathbf{E}(\mathbf{P})$ *for any pair of pants* $\mathbf{P}$.
- *of* $B_{\mathbf{P}}^b \in \mathbf{E}(\mathbf{P})$ *for any pair of pants* P *in which some* $b \in \pi_0(\partial_+ P)$ *has been selected.*
- *of* $D_{\mathbf{T}} \in \mathbf{E}(\mathbf{T})$ *for any torus* $\mathbf{T}$ *with one boundary component.*

It is enough to make such choices $(A_{\mathbf{P}}, B_{\mathbf{P}}^b, C_{\mathbf{P}}, D_{\mathbf{T}})$ for a reference $\mathbf{P}$ and $\mathbf{T}$, in such a way that

$$A_{\mathbf{P}}, C_{\mathbf{P}} \in \mathbf{E}(\mathbf{P})^{\Gamma(\mathbf{P})}, \qquad D_{\mathbf{T}} \in \mathbf{E}(\mathbf{T})^{\Gamma(\mathbf{T})},$$

and such that, for any $\varphi \in \Gamma(\mathbf{P})$ which induces a permutation $\tilde{\varphi}$ of $\pi_0(\partial_+ P)$, we have that

$$\varphi(B_{\mathbf{P}}^b) = B_{\mathbf{P}}^{\tilde{\varphi}(b)}.$$

Then, if $\mathbf{P}'$ is another pair of pants and $X \in \mathbf{E}(\mathbf{P})$, we can use a morphism $\varphi : \mathbf{P} \to \mathbf{P}'$ in $\mathrm{Bord}_1^{\bullet}$ to transport $X' = \varphi(X) \in \mathbf{E}(\mathbf{P}')$. If we take X among $A_{\mathbf{P}}, B_{\mathbf{P}}^b, C_{\mathbf{P}}$, the mapping class group invariances of these elements imply that X' is independent of φ, and indeed contained in the mapping class group invariant subspace – in the case of $B_{\mathbf{P}}^b$ this of course means that we only consider mapping classes which preserve the choice of the selected component $\varphi(b)$ in $\partial_+ P'$. The same argument holds for tori with one boundary component.

Definition 3.5 *The initial data is called* admissible *if* $A_{\mathbf{P}} \in E'(\mathbf{P})$, $D_{\mathbf{T}} \in E'(\mathbf{T})$ *and furthermore it satisfies the following axiom.*

DECAY AXIOM. *For any connected object* $\boldsymbol{\Sigma}$ *in* $\mathrm{Bord}_1^{\bullet}$, *and any* $c \in \mathfrak{C}(\boldsymbol{\Sigma})$, *we require that for any* $s > 0$, *for any* $(i,j) \in \mathscr{I}_{\mathbf{P}_c} \times \mathscr{I}_{\boldsymbol{\Sigma}_c}$ *and* $k \in \mathscr{I}_{\boldsymbol{\Sigma}}$ *such that* $k \leq h_c(i,j)$, *any* $\alpha \in \mathscr{A}_{\boldsymbol{\Sigma}}^{(k)}$, *there exists a functorial* $M_{i,s}(\Sigma) > 0$ *such that*

- *if* P_c *shares two boundary components with* Σ, *and among those,* b *denotes the one in* $\partial_+ P_c$, *then*

$$\forall v \in E'(\boldsymbol{\Sigma}_c)^{\Gamma(\boldsymbol{\Sigma}_c)} \qquad \left|\Theta_c^{i,j,k}(B_{\mathbf{P}_c}^{b}, v)\right|_{\alpha} \leq M_{i,s}(\Sigma)\, \|v\|_j\, |l_\alpha(\gamma_c)|^{-s}. \tag{5}$$

- *if* P_c *shares only one boundary component with* Σ, *then*

$$\forall v \in E'(\boldsymbol{\Sigma}_c)^{\Gamma(\boldsymbol{\Sigma}_c)} \qquad \left|\Theta_c^{i,j,k}(C_{\mathbf{P}_c}, v)\right|_{\alpha} \leq M_{i,s}(\Sigma)\, \|v\|_j \big(|l_\alpha(\gamma_c^1)| + |l_\alpha(\gamma_c^2)|\big)^{-s}. \tag{6}$$

3.3 Definition of the geometric recursion

Let (A, B, C, D) be an admissible initial data for a target theory $\mathbf{E}$.

Definition 3.6 *We define the GR amplitudes as follows. We put* $\Omega_{\varnothing} := 1 \in E(\varnothing) = \mathbb{K}$.

If $\mathbf{P}$ *is a pair of pants, we put* $\Omega_P := A_{\mathbf{P}}$.

If $\boldsymbol{\Sigma}$ *is a torus with one boundary, we put* $\Omega_{\mathbf{T}} := D_{\mathbf{T}}$.

If $\boldsymbol{\Sigma}$ *is a connected object of* $\mathrm{Bord}_1^{\bullet}$ *with Euler characteristic* $\chi(\Sigma) \leq -2$ *and let* $b_1 = \partial_- \Sigma$ *be the distinguished boundary. We now seek to inductively define*

$$\Omega_{\boldsymbol{\Sigma}} := \frac{1}{2} \sum_{c \in \mathfrak{C}_{b_1,b_1}(\boldsymbol{\Sigma})} \boldsymbol{\Theta}_c(C_{\mathbf{P_c}}, \Omega_{\boldsymbol{\Sigma}_c}) + \sum_{b \in \pi_0(\partial_+ \Sigma)} \sum_{c \in \mathfrak{C}_{b_1,b}(\boldsymbol{\Sigma})} \boldsymbol{\Theta}_c(B_{\mathbf{P}_c}^{b}, \Omega_{\boldsymbol{\Sigma}_c}) \tag{7}$$

as an element of $E(\boldsymbol{\Sigma})$.

For disconnected objects $\boldsymbol{\Sigma}$, *we declare*

$$\Omega_{\boldsymbol{\Sigma}} := \bigsqcup_{a \in \pi_0(\Sigma)} \Omega_{\boldsymbol{\Sigma}(a)}.$$

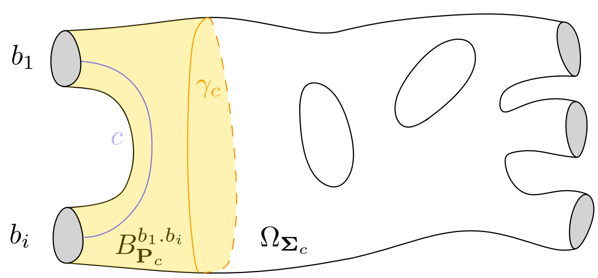

Figure 1: The B orbits (case **I'**).

Theorem 3.7 *The assignment* $\boldsymbol{\Sigma} \mapsto \Omega_{\boldsymbol{\Sigma}}$ *is well-defined. More precisely, the series converges absolutely for any of the seminorms* $|\cdot|_\alpha$, *its limit is an element of* $E'(\boldsymbol{\Sigma})$, *and it is functorial. In particular,* $\Omega_{\boldsymbol{\Sigma}}$ *is mapping class group invariant.*

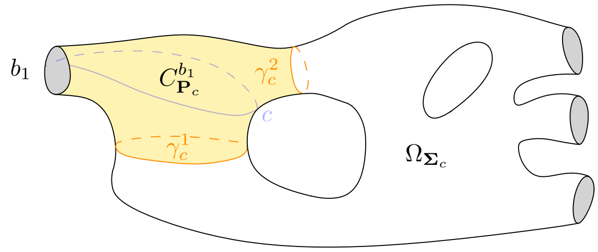

Figure 2: The C orbits (case **I**).

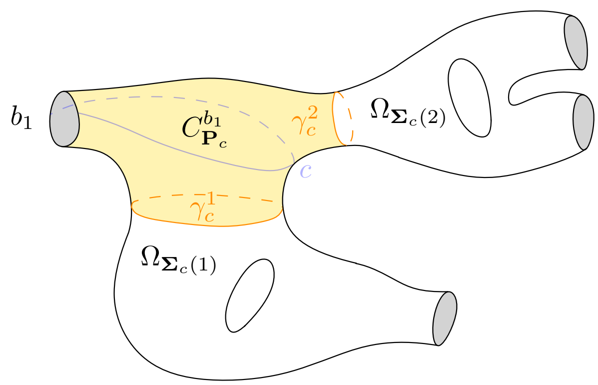

Figure 3: The C orbits (case **II**).

Proof. The result holds true in the case $\chi(\Sigma) = -1$ by the assumptions on initial data. Assume it holds for all connected objects of Euler characteristic strictly larger that a given $\chi_0 \leq -2$. Let $\mathbf{\Sigma}$ be an object in $\mathrm{Bord}_1^\bullet$ with Euler characteristic χ_0. For any $c \in \mathfrak{C}(\mathbf{\Sigma})$, the induction hypothesis applies to $\mathbf{\Sigma}_c$, in particular $\Omega_{\mathbf{\Sigma}_c}$ is $\Gamma(\mathbf{\Sigma}_c)$-invariant. Then, the discussion we had at the end of Section 3.1 shows that, for any c' in a $\Gamma(\mathbf{\Sigma})$-orbit $\mathcal{O} \subseteq \mathfrak{C}(\mathbf{\Sigma})$, we can write

$$\mathbf{\Theta}_{c'}(X, \Omega_{\mathbf{\Sigma}_{c'}}) = \mathbf{\Theta}_{c',\mathcal{O}}(X, \Omega_{\mathcal{O}})$$

for the same element $(X, \Omega_{\mathcal{O}}) \in \mathbf{E}_{\mathcal{O}}$. Here $X = B^b_{\mathbf{P}}$ or $C_{\mathbf{P}}$ depending of the nature of the orbit and $\mathbf{P}$ is a reference pair of pants in that orbit.

We first work with the unique orbit $\mathcal{O}^b = \mathfrak{C}_{b_1,b}(\mathbf{\Sigma})$ for some $b \neq b_1$, for which $X = B^b$. To have lighter notations, we just write $\mathcal{O}$ in the following argument. By the decay axiom, we have for any $s > 0$, any $(i,j) \in \mathscr{I}_{\mathbf{P}} \times \mathscr{I}_{\mathbf{\Sigma}_c}$ and $k \in \mathscr{I}_{\mathbf{\Sigma}}$, any $\alpha \in \mathscr{A}^{(k)}_{\mathbf{\Sigma}}$ that

$$\sum_{c' \in \mathcal{O}} \left|\Theta^{i,j,k}_{c'}(B^b_{\mathbf{P}_{c'}}, \Omega_{\mathcal{O}})\right|_\alpha \leq m_k M_{i,s} \|\Omega_{\mathcal{O}}\|_j \, \zeta_\alpha(s; \mathcal{O}) \,, \tag{8}$$

where

$$\zeta_\alpha(s; \mathcal{O}) = \sum_{\gamma \in \mathcal{O}} (l_\alpha(\gamma))^{-s} \in (0, +\infty] \,.$$

The polynomial growth axiom implies that $\zeta_\alpha(s; \mathcal{O})$ is finite for any $s > d_k(\Sigma)$, and the lower bound axiom further implies there exists a finite constant M'_k such that for the choice $s_k = d_k(\Sigma) + 1$ we have that

$$\sup_{\alpha \in \mathscr{A}^{(k)}_{\mathbf{\Sigma}}} \zeta_\alpha(s_k; \mathcal{O}) \leq M'_k \,. \tag{9}$$

Thus we get that

$$\sum_{\gamma\in\mathcal{O}} \left|\Theta_\gamma^{i,j,k}(B^b_{P_\gamma},\Omega_{\mathcal{O}})\right|_\alpha \le m_k M_{i,s_k}\,\|\Omega_{\mathcal{O}}\|_j\, M'_k. \tag{10}$$

This proves that the series $\sum_{\gamma\in\mathcal{O}} \Theta_\gamma^{i,j,k}(\mathbf{B}^b_{P_\gamma},\Omega_{\mathcal{O}})$ is absolutely convergent in $E^{(k)}(\mathbf{\Sigma})$. Let us denote $w_{\mathcal{O}}^{(i,j,k)}$ the limit. The inequality (10) implies

$$\|w_{\mathcal{O}}^{(i,j,k)}\|_k = \sup_{\alpha\in\mathscr{A}^{(k)}(\mathbf{\Sigma})} |w_{\mathcal{O}}^{(i,j,k)}|_\alpha \le m_k M_{i,s_k}\,\|\Omega_{\mathcal{O}}\|_j\, M'_k. \tag{11}$$

Since all involved maps are compatible with projective limits, so are $w_{\mathcal{O}^b}^{(i,j,k)}$ which also means it only depends on $k\in\mathscr{I}_{\mathbf{\Sigma}}$

$$w_{\mathcal{O}}^{(k)} = w_{\mathcal{O}}^{(i,j,k)}$$

and we get a well-defined, unique element

$$\Omega_{\mathbf{\Sigma}}(B^b;\mathcal{O}) \coloneqq \mathbf{\Theta}_{c,\mathcal{O}}(B^b_{\mathbf{P}},\Omega_{\mathcal{O}}) \in E(\mathbf{\Sigma}),$$

which according to (11) in fact belongs to the subspace $E'(\mathbf{\Sigma})$.

The same reasoning for the orbit of a $c\in\mathfrak{C}_{b_1,b_1}(\mathbf{\Sigma})$ – which corresponds to a C – shows as well that we have a well-defined, unique element

$$\Omega_{\mathbf{\Sigma}}(C;\mathcal{O}) \coloneqq \mathbf{\Theta}_{c,\mathcal{O}}(C_{\mathbf{P}},\Omega_{\mathcal{O}}) \in E'(\mathbf{\Sigma}).$$

If we let $b_1 = \partial_-\Sigma$ and sum over the finitely many distinct $\Gamma(\Sigma)$-orbits, we thus have a well-defined element

$$\begin{aligned}\Omega_{\mathbf{\Sigma}} &\coloneqq \sum_{b\in\pi_0(\partial_+\Sigma)} \Omega_{\mathbf{\Sigma}}(B^b;\mathcal{O}^b) + \tfrac{1}{2}\sum_{\mathcal{O}\in\mathfrak{C}_{b_1,b_1}(\mathbf{\Sigma})/\Gamma(\mathbf{\Sigma})} \Omega_{\mathbf{\Sigma}}(C;\mathcal{O}) \\ &= \sum_{b\in\pi_0(\partial_+\Sigma)} \mathbf{\Theta}_c(B^b_{\mathbf{P}_c},\Omega_{\mathbf{\Sigma}_c}) + \tfrac{1}{2}\sum_{c\in\mathfrak{C}_{b_1,b_1}(\mathbf{\Sigma})} \mathbf{\Theta}_c(C_{\mathbf{P}_c},\Omega_{\mathbf{\Sigma}_c}).\end{aligned}$$

Let $\varphi:\mathbf{\Sigma}\to\mathbf{\Sigma}'$ be a morphism in $\mathrm{Bord}_1^\bullet$. For $c\in\mathfrak{C}(\mathbf{\Sigma})$, we get $\varphi(c)\in\mathfrak{C}(\mathbf{\Sigma}')$ and φ induces a morphism $\tilde\varphi$ in $\mathrm{Bord}_1^\bullet$ between $\mathbf{P}_c\cup\mathbf{\Sigma}_c$ and $\mathbf{P}'_{\varphi(c)}\cup\mathbf{\Sigma}'_{\varphi(c)}$. The compatibility of the glueing maps with glueing of morphisms implies that

$$\mathbf{E}(\varphi)\big(\mathbf{\Theta}_c(X_{\mathbf{P}_c},\Omega_{\mathbf{\Sigma}_c})\big) = \mathbf{\Theta}_{\varphi(c)}\big(\mathbf{E}(\tilde\varphi)(X_{\mathbf{P}_c},\Omega_{\mathbf{\Sigma}_c})\big).$$

We apply this formula to $X_{\mathbf{P}_c} = B^b_{\mathbf{P}_c}$ or $C_{\mathbf{P}_c}$, which are functorial initial data. The induction hypothesis guarantees that $\Omega_{\mathbf{\Sigma}_c}$ is also functorial. Therefore,

$$\mathbf{E}(\varphi)\big(\mathbf{\Theta}_c(X_{P_c},\Omega_{\mathbf{\Sigma}_c})\big) = \mathbf{\Theta}_{\varphi(c)}\big(X_{\mathbf{P}'_{\varphi(c)}},\Omega_{\mathbf{\Sigma}'_{\varphi(c)}}\big).$$

This gives a bijection between the terms in $\mathbf{E}(\varphi)(\Omega_{\mathbf{\Sigma}})$ and the terms in $\Omega_{\mathbf{\Sigma}'}$, implying, thanks to absolute convergence for each seminorm, that

$$(\varphi)(\Omega_{\mathbf{\Sigma}}) = \Omega_{\mathbf{\Sigma}'}\,,$$

and thus functoriality. In particular, taking $\mathbf{\Sigma}' = \mathbf{\Sigma}$ gives the mapping class group invariance $\mathbf{E}(\varphi)(\Omega_{\mathbf{\Sigma}}) = \Omega_{\mathbf{\Sigma}}$. We conclude the proof by induction. ∎

Remark: weakening admissibility

The notion of admissibility in Definition 3.5 is a sufficient condition for the result of Theorem 3.7 to hold, but is not optimal. This condition implies that the convergence of $\Omega_{\mathbf{\Sigma}}$ is controlled by the convergence of the zeta function of the length spectra, which is itself guaranteed by the polynomial growth axiom on length functions. It is sometimes possible and desirable to find weaker conditions which imply Theorem 3.7. The key point is, for any stable $\mathbf{\Sigma}$, to secure the absolute convergence of the series in (7), and an upper bound showing that the outcome belongs to the subspace $E'(\mathbf{\Sigma})$ defined in (4).

We will see in Section 10.2 an example of weaker admissibility condition following from the Mirzakhani-McShane identities.

3.4 Natural transformations of target theories

Let $\mathbf{1} : \mathrm{Bord}_1^{\bullet} \to \mathcal{C}$ be the symmetric monoidal functor which assigns the unit object $\mathbb{K}$ to any object $\mathbf{\Sigma}$ in $\mathrm{Bord}_1^{\bullet}$. A functorial assignment in $\mathbf{E}(\Sigma)$ is a natural transformation from the functor $\mathbf{1}$ to the functor $\mathbf{E}$. For any admissible data, the GR amplitude provides such a natural transformation.

More generally, let $\mathbf{E}$ and $\tilde{\mathbf{E}}$ be two target theories, and $\eta : \mathbf{E} \Rightarrow \tilde{\mathbf{E}}$ be a natural transformation compatible with the union the glueing morphisms, and the length functions. If

$$\mathcal{I} := (A_P, B_P^b, C_P, D_T)$$

are admissible initial data for $\mathbf{E}$, it is clear that

$$\eta(\mathcal{I}) := (\eta_P(A_P), \eta_P(B_P^b), \eta_P(C_P), \eta_T(D_T))$$

are admissible initial data for $\tilde{\mathbf{E}}$. Denote $\Omega^{\mathcal{I}}$ and $\tilde{\Omega}^{\eta(\mathcal{I})}$ the $\mathbf{E}$-valued (respectively $\tilde{\mathbf{E}}$-valued) outcome of GR from these two set of initial data.

Proposition 3.8 *For any object* $\mathbf{\Sigma}$ *in* $\mathrm{Bord}_1^{\bullet}$, $\tilde{\Omega}_{\mathbf{\Sigma}}^{\eta(\mathcal{I})} = \eta_{\mathbf{\Sigma}}(\Omega_{\mathbf{\Sigma}}^{\mathcal{I}})$.

Proof. This is directly implied by the fact that $\eta_{\mathbf{\Sigma}} : \mathbf{E}(\Sigma) \to \tilde{\mathbf{E}}(\Sigma)$ is continuous, linear, and that the assignment $\mathbf{\Sigma} \mapsto \eta_{\mathbf{\Sigma}}$ is compatible with union and glueing maps and length functions. ∎

3.5 Sums over embedded fatgraphs

In this paragraph, we show that the geometric recursion can be repackaged as a (finite sum) over fatgraphs of the appropriate topology with no reference to embeddings. The contribution of each fatgraph also satisfy a recursion when the first vertex is removed, albeit of non local nature. The similarity but also important differences with the topological recursion of [26] will be spelled out in Section 6.

Preliminary: fatgraphs and their doubled surface

In this paragraph we assumed $2g - 2 + n > 1$. We consider the set $\mathbb{G}^{g,n}$ of connected uni-trivalent fatgraphs with exactly n univalent vertices and genus g. We denote by $\mathbb{G}_1^{g,n}$ the set of graphs G in $\mathbb{G}^{g,n}$ equipped with a distinguished univalent vertex, called the *root*.

We recall the construction of the *canonical spanning tree* $\mathfrak{t}(G)$ for any element $G \in \mathbb{G}_1^{g,n}$ (see Figure 4 for an example). We inductively define $\mathfrak{t}(G)$ by starting at the root in the direction of its incident edge, traveling on the edges of G respecting the cyclic order of the half-edges at vertices, adding at each step to $\mathfrak{t}(G)$ the edge we encounter along the boundary component we are traveling on, whenever it has not already been added and it does not create a loop. Once we come back to the root, we restart the same travel until we get to the first edge e, for which the boundary component on the other side of the edge is different from the one we are traveling on currently. We then jump to this next boundary component and continue the travel along this component, adding edges to $\mathfrak{t}(G)$ with the same rules. When we come back to the starting point of travel for this boundary component next to edge e, we iterate the process (travel till we can jump to an unexplored opposite boundary component), until we cannot add more edges without creating a loop. The outcome is a rooted tree $\mathfrak{t}(G)$, which spans all the vertices of G.

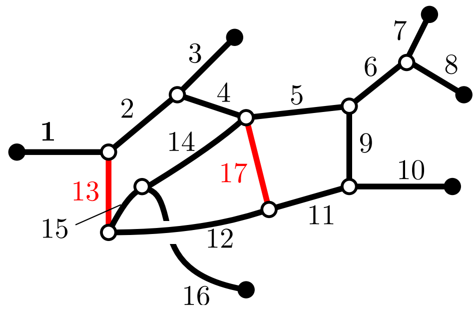

Figure 4: The exploration on a fatgraph $G \in \mathbb{G}_1^{2,6}$. We start from the root **1**, and the numbers indicate in which order we then meet the (unoriented) edges. The spanning tree $\mathfrak{t}(G)$ is in black.

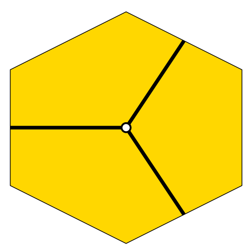

Figure 5: Hexagon attached to a trivalent vertex.

To any $G \in \mathbb{G}_{g,n}$ one can associate a surface $\Sigma(G)$ of genus g with n boundary components, called the *double* (Figure 6). Let us briefly recall this construction. For each vertex of v, we take two copies H_v^+ and H_v^- of a closed hexagon in which v and its incident half-edges are embedded such that they end on three (alternating) sides of the hexagon (Figure 5). If there is an edge from a univalent vertex u to v, we ask that the edge from u to v is embedded in $H_v^{\pm}$ such that u maps to a point in the interior of a side of the hexagon. We orient H_v^+ (respectively H_v^-) so as to agree (respectively disagree) with the cyclic order of half-edges around v. We then glue together H_v^+ and H_v^- into a pair of pants P_v by identifying pairs (s_-, s_+) of sides with opposite orientation in (H_v^-, H_v^+) which are not crossed by edges incident to v. P_v is equipped with a marked point on each of its boundary component, remembering where the edges intersect the sides of H_v^+. The surface $\boldsymbol{\Sigma}(G)$ is then obtained by the quotient of $\bigcup_v P_v$, where we for each edge in G, say between the vertices v and v', identify the pointed boundary components of P_v and $P_{v'}$ corresponding to the edge. In particular, two copies of G are embedded in $\boldsymbol{\Sigma}(G)$, and the image $\Sigma^+(G)$ of $\bigcup_v H_v^+$ in $\Sigma(G)$ deformation retracts onto G.

Since two different ways of choosing the topological spaces $H_v^{\pm}$ and the glueing are related by diffeomorphisms in a unique orientation-preserving isotopy class, $\boldsymbol{\Sigma}(G)$ can be considered as a single object in $\mathrm{Bord}^{\bullet}$ up to canonical isomorphisms.

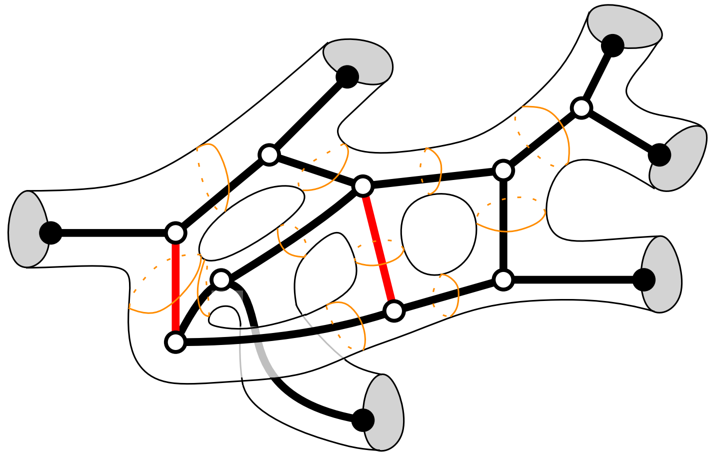

Figure 6: The double of the graph of Figure 4
.

If $G \in \mathbb{G}_1^{g,n}$, reversing the orientation of the boundary component of $\Sigma(G)$ containing the root gives a unique object in $\mathrm{Bord}_1^\bullet$ up to canonical isomorphisms. In the remaining G always stands for a graph in $\mathbb{G}_1^{g,n}$ and $\boldsymbol{\Sigma}(G)$ such an object in $\mathrm{Bord}_1^\bullet$.

Let $\widetilde{\mathbb{M}}_G(\boldsymbol{\Sigma})$ be the set of morphisms from $\boldsymbol{\Sigma}(G)$ to $\boldsymbol{\Sigma}$ such that the univalent vertices of G are mapped to the marked points on $\partial\boldsymbol{\Sigma}$, and the root is mapped to the marked point on $\partial_-\boldsymbol{\Sigma}$. We observe that an element of $\widetilde{\mathbb{M}}_G(\boldsymbol{\Sigma})$ induces a pair of pants decomposition of $\boldsymbol{\Sigma}$, up to ambient isotopy. Thus there is a well-defined subgroup Γ_G associated to each G, which is the subgroup of $\Gamma(\boldsymbol{\Sigma})$ preserving each pair of pant, as well as each boundary component of a pair of pants in this decomposition. We denote the quotient by

$$\mathbb{M}_G(\boldsymbol{\Sigma}) \coloneqq \widetilde{\mathbb{M}}_G(\boldsymbol{\Sigma})/\Gamma_G,$$

and we will write

$$\widetilde{\mathbb{M}}(\boldsymbol{\Sigma}) \coloneqq \bigcup_{G \in \mathbb{G}_1^{g,n}} \widetilde{\mathbb{M}}_G(\boldsymbol{\Sigma}), \qquad \mathbb{M}(\boldsymbol{\Sigma}) \coloneqq \bigcup_{G \in \mathbb{G}_1^{g,n}} \mathbb{M}_G(\boldsymbol{\Sigma}).$$

GR as a sum over fatgraphs

The spanning tree $\mathfrak{t}(G)$ of G allows us to determine an ordering on the embedded pair of pants $(P_1, \dots P_{2g-2+n})$. We put $\boldsymbol{\Sigma}(G)_1 \coloneqq \boldsymbol{\Sigma}(G)$, and successively excise the pair of pants $\mathbf{P}_i$, for $i = 1, 2, \dots$ we obtain a sequence of surfaces inductively defined by $\boldsymbol{\Sigma}(G)_{i+1} = \boldsymbol{\Sigma}(G)_i - \mathbf{P}_i$. When excising $\mathbf{P}_i$, we can distinguish the following four mutually exclusive cases.

A) $\partial\mathbf{P}_i \subseteq \partial\boldsymbol{\Sigma}(G)_i$.

B) Exactly two of the boundary components of $\mathbf{P}_i$ are boundary components of $\boldsymbol{\Sigma}(G)_i$.

C) Exactly one of the boundary components of $\mathbf{P}_i$ is a boundary component of $\Sigma(G)_i$, and the two other boundary components of P_i are disjointly embedded in $\boldsymbol{\Sigma}(G)_i$.

D) Two of the boundary components of $\mathbf{P}_i$ are glued together under the embedding into $\boldsymbol{\Sigma}(G)_i$. In this case, we rather replace $\mathbf{P}_i$ by a torus (which we keep denoting $\mathbf{P}_i$ for convenience) with one boundary obtained by glueing these two boundary components.

This defines for us a type map $X : [\![1, 2g-2+n]\!] \to \{A, B, C, D\}$.

In a given target theory $\mathbf{E}$, let us denote

$$\mathbf{\Theta}_G : \prod_{i=1}^{2g-2+n} \mathbf{E}(\mathbf{P}_i) \to \mathbf{E}(\mathbf{\Sigma}(G))$$

the glueing morphism, and for given admissible initial data, we define $\omega_G \in \mathbf{E}(\mathbf{\Sigma}(G))$ by

$$\omega_G := \mathbf{\Theta}_G\big((X_{\mathbf{P}_i}(i))_{i=1}^{2g-2+n}\big).$$

For an $\tilde{\varphi} \in \widetilde{\mathbb{M}}(\mathbf{\Sigma})$ we have of course a morphism $\mathbf{E}(\tilde{\varphi}) : \mathbf{E}(\mathbf{\Sigma}(G)) \to \mathbf{E}(\mathbf{\Sigma})$. Since the mapping classes in Γ_G preserve individually each boundary components of the $\mathbf{P}_i$s, ω_G is Γ_G-invariant. Therefore $\mathbf{E}(\tilde{\varphi})(\omega_G) \in \mathbf{E}(\mathbf{\Sigma})$ only depends on the projection $\varphi \in \mathbb{M}(\mathbf{\Sigma})$ of $\tilde{\varphi}$, and can be denoted $\mathbf{E}(\varphi)(\omega_G)$.

Proposition 3.9 *For any object $\mathbf{\Sigma}$ in* $\mathrm{Bord}_1^{\bullet}$*, the GR amplitudes satisfy*

$$\Omega_{\mathbf{\Sigma}} = \sum_{G \in \mathbb{G}_1^{g,n}} \Omega_{\mathbf{\Sigma}}^G, \qquad \Omega_{\mathbf{\Sigma}}^G := \sum_{\varphi \in \mathbb{M}_G(\mathbf{\Sigma})} \mathbf{E}(\varphi)(\omega_G).$$

Proof. Unfolding the recursive definition (7) and thanks to the property of absolute convergence from Theorem 3.7, we can apply the Fubini theorem and obtain a sum over sequences $(c_1, \ldots, c_m)$ where $c_i \in \mathfrak{C}(\mathbf{\Sigma}_{c_1,\ldots,c_{i-1}})$, with $\mathbf{\Sigma}_{\varnothing} := \mathbf{\Sigma}$ and $\mathbf{\Sigma}_{c_1,\ldots,c_i} = \mathbf{\Sigma}_{c_1,\ldots,c_{i-1}} - \mathbf{P}_{c_i}$. At each step, the excision of a pair of pants produces the isotopy class of embeddings $f_i : \mathbf{P}_{c_i} \to \mathbf{\Sigma}_{c_1,\ldots,c_{i-1}}$ of a pair of pants with ordered boundary components. What is left, when no more excisions are possible (recall we do not allow $\mathbf{\Sigma}_c = \varnothing$), are connected components which are either pairs of pants (an A factor) or tori with one boundary component (a D factor). Tracking the ordering of boundary component of the $\mathbf{P}_{c_i}$ in this nested structure reveals that this set of sequences is in bijection with $\mathbb{M}(\Sigma)$. For each $\varphi \in \mathbb{M}(\mathbf{\Sigma})$, we can then rearrange the nested application of glueing maps into a single glueing map $\mathbf{E}(\varphi) \circ \mathbf{\Theta}_G$ using associativity of glueings in the target theory. It should be applied to a tuple formed by the factors contributing to each embedded piece $\mathbf{P}_i$, either an A, B, C or D, depending only on the abstract fatgraph G as listed above. ∎

FATGRAPH RECURSION

We can also write a fatgraph recursion, but we stress that the glueing maps will be non local, *i.e.* they *a priori* not only depend on the excision of a trivalent vertex, but on the full structure of the fatgraph. This makes our recursion quite different from usual fatgraph recursions encountered in enumerative geometry, as we will comment on in Section 6.

Let us define $\mathbf{E}_G := \mathbf{E}(\Sigma(G))$ and let

$$\Omega_G := \sum_{\varphi \in \mathbb{M}_G(\mathbf{\Sigma}(G))} \mathbf{E}(\varphi)(\omega_G) \in \mathbf{E}_G. \tag{12}$$

Let $\mathfrak{t}_1 \subset \mathfrak{t}(G)$ be the union of the univalent vertex u, the trivalent vertex v incident to it, and the half edges attached to these two vertices. $\tilde{G} := G - \mathfrak{t}_1$ may or may not be a connected graph. If it is connected we put $\tilde{\mathbf{E}}_{\tilde{G}} := \mathbf{E}_{\tilde{G}}$. If it is not connected, $\tilde{G}$ will consist of an ordered (by cyclic order at v starting with $u \to v$) pair of connected components $(\tilde{G}_1, \tilde{G}_2)$, and we define $\tilde{\mathbf{E}}_{\tilde{G}} := \mathbf{E}_{\tilde{G}_1} \cup \mathbf{E}_{\tilde{G}_2}$. In both cases, we construct a *fatgraph glueing map*

$$\mathbf{\Theta}_{G,\mathfrak{t}_1} : \mathbf{E}_{\mathfrak{t}_1} \times \tilde{\mathbf{E}}_{G-\mathfrak{t}_1} \to \mathbf{E}_G$$

as follows. Let $\mathbb{M}(G, \mathfrak{t}_1)$ be the set of all embeddings of the pair of pants $\mathbf{\Sigma}(\mathfrak{t}_1)$ into $\mathbf{\Sigma}(G)$ up to isotopy, only preserving the pointed negative boundary component. For each $p \in \mathbb{M}(G, \mathfrak{t}_1)$ we choose an identification φ_p of the complement of p with $\mathbf{\Sigma}(G - \mathfrak{t}_1)$ in such a way that the pointed negative boundary components are preserved. This way we obtain glueing map $\mathbf{\Theta}_p : \mathbf{E}_{\mathfrak{t}_1} \times \tilde{\mathbf{E}}_{\tilde{G}} \to \mathbf{E}_G$, which does not depend on the choice of φ_p, and we define

$$\Theta_{G,\mathfrak{t}_1} := \sum_{p \in \mathbb{M}(G,\mathfrak{t}_1)} \mathbf{\Theta}_p \,. \tag{13}$$

This series of maps from $\mathbf{E}_{\mathfrak{t}_1} \times \tilde{\mathbf{E}}_{\tilde{G}}$ to $\mathbf{E}_G$ converges absolutely in the topology of pointwise convergence according to Lemma 3.7 and the convergence axiom for initial data.

Proposition 3.10 *For any $G \in \mathbb{G}_1^{g,n}$ with $2g - 2 + n \geq 2$, we have $\Omega_G = \mathbf{\Theta}_{G,\mathfrak{t}_1}(\Omega_{\mathfrak{t}_1}, \Omega_{G-\mathfrak{t}_1})$.*

Proof. As the series (12) is absolutely convergent, it can be repackaged as

$$\Omega_G = \sum_{p \in \mathbb{M}(G,\mathfrak{t}_1)} \sum_{\substack{\varphi \in \mathbb{M}_G(\mathbf{\Sigma}(G)) \\ \varphi|_{\mathfrak{t}_1} = p}} \mathbf{E}(\varphi)(\omega_G).$$

By the definition of $\mathbf{\Theta}_{G,\mathfrak{t}_1}$, $\Omega_{G-\mathfrak{t}_1}$ and Ω_G, this sum may be rewritten as $\Omega_G = \mathbf{\Theta}_{G,\mathfrak{t}_1}(\Omega_{\mathfrak{t}_1}, \Omega_{G-\mathfrak{t}_1})$. ■

4 Three further aspects

In this section, we elaborate some aspects of GR which can be skipped in a first reading.

4.1 Control for large boundary lengths

It can be useful to have some control for the GR amplitudes for large boundary lengths – in terms of the length functions provided by the target theory. We introduce a finer notion of admissibility for initial data which allow us to do so. In this paragraph, $\daleth : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is a given function which encodes the type of growth we may allow. Let $\mathbf{E} : \mathrm{Bord}_1^{\bullet} \to \mathcal{C}$ be a target theory and (A, B, C, D) some initial data.

For any object $\mathbf{\Sigma}$, we define modified (possibly infinite) seminorms on $E(\mathbf{\Sigma})$ indexed by $i \in \mathscr{A}_{\mathbf{\Sigma}}^{(i)}$

$$\|v\|_i^{\daleth} := \sup_{\alpha \in \mathscr{A}_{\mathbf{\Sigma}}^{(i)}} \frac{|v|_\alpha}{\prod_{\beta \in \pi_0(\partial\Sigma)} \daleth\big(|l_\alpha(\beta)|\big)} \in [0, +\infty] \tag{14}$$

and the (possibly not closed) subspace

$${}^{\daleth}E'(\mathbf{\Sigma}) := \Big\{ v \in E(\mathbf{\Sigma}) \quad \Big| \quad \forall i \in \mathscr{I}_{\mathbf{\Sigma}} \quad \|v\|_i^{\daleth} < +\infty \Big\} \,. \tag{15}$$

Definition 4.1 *The initial data (A, B, C, D) is called admissible with $\daleth$-decay if $A_{\mathbf{P}} \in E'(\mathbf{P})$, $D_{\mathbf{T}} \in E'(\mathbf{T})$ and furthermore it satisfies the following axiom.*

$\daleth$-DECAY AT INFINITY. *For any connected object $\mathbf{\Sigma}$ in $\mathrm{Bord}_1^{\bullet}$, for any $c \in \mathfrak{C}(\mathbf{\Sigma})$, we require that for any $s > 0$, $(i, j) \in \mathscr{I}_{\mathbf{P}_c} \times \mathscr{I}_{\mathbf{\Sigma}_c}$ and $k \in \mathscr{I}_{\mathbf{\Sigma}}$ such that $k \leq h_c(i, j)$, there exists a functional $M_{i,s}^{\daleth}(\Sigma) > 0$ such that*

- *if P_c shares two boundary components with Σ, namely $b_1 = \partial_-\Sigma$ and $b \in \partial_+\Sigma$, and $\gamma_c \subset \Sigma^\circ$ is the last boundary component of P_c, then for any $v \in E'(\mathbf{\Sigma}_c)^{\Gamma(\mathbf{\Sigma}_c)}$*

$$\sup_{\alpha \in \mathscr{A}_{\mathbf{\Sigma}_c}^{(k)}} \left\{ \frac{\daleth\big(|l_\alpha(\gamma_c)|\big)\, |l_\alpha(\gamma_c)|^s}{\daleth\big(|l_\alpha(b_1)|\big)\daleth\big(|l_\alpha(b)|\big)} \left|\Theta_c^{i,j,k}(B_{\mathbf{P}_c}^b, v)\right|_\alpha \right\} \leq M_{i,s}^{\daleth}(\Sigma) \|v\|_j^{\daleth}. \tag{16}$$

- *if P_c shares only one boundary component with Σ, then for any $v \in E'(\mathbf{\Sigma}_c)^{\Gamma(\mathbf{\Sigma}_c)}$*

$$\sup_{\alpha \in \mathscr{A}^{(k)}_{\mathbf{\Sigma}}} \left\{ \frac{\daleth\big(|l_\alpha(\gamma^1_c)|\big)\daleth\big(|l_\alpha(\gamma^2_c)|\big)}{\daleth\big(|l_\alpha(b_1)|\big)} \big(|l_\alpha(\gamma^1_c)| + |l_\alpha(\gamma^2_c)|\big)^s \left|\mathbf{\Theta}^{i,j,k}_c(C_{\mathbf{P}_c}, v)\right|_\alpha \right\} \leq M^{\daleth}_{i,s}(\Sigma)\, \|v\|^{\daleth}_j.$$

We note that the decay axiom in Definition 3.5 is equivalent to the $\daleth$-decay at ∞ for the choice $\daleth = 1$. Due to the polynomial growth and lower bound axioms for lengths, what really matters in this definition is the behaviour of $\daleth$ at ∞.

Proposition 4.2 *Assume (A, B, C, D) are initial data which are admissible with $\daleth$-decay. Then, GR from Definition 3.6 gives a well-defined, functorial assignment $\mathbf{\Sigma} \mapsto \Omega_{\mathbf{\Sigma}} \in {}^{\daleth}E'(\mathbf{\Sigma})$.*

Proof. We follow the proof of Theorem 3.7. The only difference is that (8) is now replaced with

$$\sup_{\alpha \in \mathscr{A}^{(k)}_{\mathbf{\Sigma}}} \left\{ \sum_{c' \in \mathcal{O}} \frac{\left|\Theta^{i,j,k}_c(B^b_{\mathbf{P}_c}, \Omega_{\mathcal{O}})\right|_\alpha}{\prod_{\beta \in \pi_0(\partial\Sigma)} \daleth\big(|l_\alpha(\beta)|\big)} \right\} \leq m_k M^{\daleth}_{i,s} \|\Omega_{\mathcal{O}}\|^{\daleth}_j \sup_{\alpha \in \mathscr{A}^{(k)}_{\mathbf{\Sigma}}} \zeta_\alpha(s; \mathcal{O})$$

since the factor $\daleth\big(|l_\alpha(\gamma)|\big)$ appearing in the left-hand side of (16) is compensated by its inverse appearing in the definition (14) of seminorm $\|\cdot\|^{\daleth}_k$. We can use (9) and obtain instead of (11)

$$\|w^{i,j,k}_{\mathcal{O}}\|^{\daleth}_k \leq m_k M_{i,s_k} \|\Omega_{\mathcal{O}}\|_j \tag{17}$$

which allows the conclusion that the limit of the series exists in $E(\mathbf{\Sigma})$ and define a unique element

$$\Omega_{\mathbf{\Sigma}}(B^b; \mathcal{O}) = \mathbf{\Theta}_{c,\mathcal{O}}(B^b_{\mathbf{P}}, \Omega_{\mathcal{O}}) \in {}^{\daleth}E'(\mathbf{\Sigma})$$

by comparison of (17) with the definition (15) of the subspace ${}^{\daleth}E'(\mathbf{\Sigma})$. The remaining of the proof of Theorem 3.7 then applies and results in the claim. ■

4.2 Inducing an initial data for tori with one boundary

In quantum field theories, defining correlation functions for tori with one boundary by a cutting procedure often involves a renormalisation procedure to get rid of infinities. We avoid to address these potential problems in the construction of target theories and admissible initial data by not requiring *a priori* in the axioms that we have self-glueings at our disposal.

Imagine that we are nevertheless given a functorial morphism,

$$\mathbf{\Xi} : \mathbf{E}(\mathbf{P}) \to \mathbf{E}(\mathbf{T})$$

where $\mathbf{P}$ is a pair of pants seen as an object in $\mathrm{Bord}^{\bullet}_1$, and $\mathbf{T}$ is the torus with one boundary obtained by glueing the two boundary components in $\partial_+ P$.

If $\mathbf{T}$ is now an arbitrary object in $\mathrm{Bord}^{\bullet}_1$ which is a torus with one boundary, we obtain a pair of pants $\mathbf{P}_\gamma$ cutting $\mathbf{T}$ along a pointed oriented simple closed curve γ which is not boundary-parallel. By definition of cutting/glueing, this $\mathbf{T}$ is obtained by self-glueing on $\mathbf{P}_\gamma$, and we denote $\mathbf{\Xi}_\gamma$ the self-glueing morphism.

Assume that we are given a functorial $\mathbf{P} \mapsto C_{\mathbf{P}} \in \mathbf{E}(\mathbf{P})$, in particular $C_{\mathbf{P}}$ must be $\Gamma(\mathbf{P})$ invariant. We remark that, due to the assumed invariance of C under braiding of the two boundary components of $\partial_+\Sigma$, and the assumed functoriality of the self-glueing morphism, $\mathbf{\Xi}_\gamma(C_{\mathbf{P}_\gamma})$ does not depend on the orientation of γ, neither on the choice of the marked point on γ. Therefore, it makes sense to consider $\gamma \in \hat{S}^\circ_\Sigma$ in these formulae. We may seek to define

$$\Omega_{\mathbf{T}} := \sum_{\gamma \in \hat{S}^\circ_T} \boldsymbol{\Xi}_\gamma(C_{\mathbf{P}_\gamma}) \in \mathbf{E}(\mathbf{T}) .$$

For this purpose, we introduce a new axiom for C.

Self-glueing C-decay. Assume that we are given a functorial self-glueing morphism for pairs of pants as above. For any object $\mathbf{T}$ in $\mathrm{Bord}_1^\bullet$ which is a torus with one boundary, for any $\gamma \in \hat{S}^\circ_T$, for any $s > 0$, any $i \in \mathscr{I}_{\mathbf{P}_\gamma}$ and $k \in \mathscr{I}_{\mathbf{T}}$ such that $k \leq h_\gamma(i)$, there exists a constant $M_{i,s}(T)$ such that for any $\alpha \in \mathscr{A}^{(k)}_{\mathbf{T}}$, we have

$$\left|\Xi^{i,k}_\gamma(C_{\mathbf{P}_\gamma})\right|_\alpha \leq M_{i,s}(T)|l_\alpha(\gamma)|^{-s} . \tag{18}$$

By an argument of absolute convergence similar to the one detailed in the proof of Theorem 3.7, we find

Lemma 4.3 *Assume (A,B,C) satisfy the properties listed in Definition 3.5 and the self-glueing decay axiom. Then*

$$D_{\mathbf{T}} = \sum_{\gamma \in \hat{S}^\circ_T} \boldsymbol{\Xi}_\gamma(C_{\mathbf{P}_\gamma}) \tag{19}$$

is a well-defined, functorial assignment for any object $\mathbf{T}$ in $\mathrm{Bord}_1^\bullet$ which is a torus with one boundary, and (A,B,C,D) is an admissible initial data.

Note that the (A,B,C) parts of the initial data are sufficient to define the GR amplitudes $\Omega_{\mathbf{\Sigma}}$ for surfaces $\mathbf{\Sigma}$ of genus 0. The D part is necessary to extend this definition to positive genus. Lemma 4.3 gives a way to induce a D if we have glueing morphisms for pairs of pants and a C satisfying the self-glueing decay axiom. Note that we could replace C by A in the discussion. Note that $B_{\mathbf{P}}$ does not a priori have the invariance under braiding of the two boundary components of $\partial_+ P$. If we insist in using a B to induce a D, we should replace (19) with the formula

$$\sum_{\gamma \in \hat{S}^{\circ,\mathrm{or}}_T} \boldsymbol{\Xi}_\gamma(B^{\gamma_+}_{\mathbf{P}_\gamma}) ,$$

where $\hat{S}^{\circ,\mathrm{or}}_T$ now enumerates oriented simple closed curves which are not boundary-parallel, and γ_+ is the boundary component of created to the left of γ when we cut $\mathbf{T}$.

Lemma 4.3 can easily be generalised to handle ¬-decay versions of (18).

Simple closed curves on tori

We review the standard description of the sets of $\hat{S}^\circ_T$ (and $\hat{S}^{\circ,\mathrm{or}}_T$) of (oriented) simple closed curves on tori with one boundary.

It is convenient to first consider $\check{\mathbf{T}}$ the marked closed torus obtained from $\mathbf{T}$ by glueing a disk along its boundary, forgetting the marked point on the (former) boundary, and adding a marked point in the interior of the disk. We also denote $\check{T}$ the closed torus obtained by further forgetting the marked point. Let us fix two oriented simple closed curve α and β in $T \subset \check{T}$ having the signed intersection number $\alpha \cap \beta = 1$. In particular, (α, β) is an integral basis for the homology of $\check{T}$. We can choose the orientation such that $\alpha\beta\alpha^{-1}\beta^{-1}$ is homologous to the (already oriented) boundary of $\mathbf{T}$. The mapping class group of $\check{T}$ is

$$\Gamma(\check{T}) \;\cong\; \left\langle s, \delta_\alpha \;\middle|\; s^4 = 1 \text{ and } s^2 = (s\delta_\alpha)^3 \right\rangle \;\cong\; \left\langle \delta_\alpha, \delta_\beta \;\middle|\; \delta_\alpha\delta_\beta^{-1}\delta_\alpha = \delta_\beta^{-1}\delta_\alpha\delta_\beta^{-1} \right\rangle \;\cong\; \mathrm{SL}(2,\mathbb{Z}) . \tag{20}$$

Here s is the flip, δ_γ the Dehn twist along γ, and these mapping class acts on the homology basis (α, β) by the $\mathrm{SL}(2,\mathbb{Z})$ matrix

$$s \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \delta_\alpha \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \delta_\beta \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

where $\equiv$ means identification via the last isomorphism in (20). In particular we have

$$s = \delta_\beta \delta_\alpha^{-1} \delta_\beta \,.$$

The canonical map $\Gamma(\check{\mathbf{T}}) \to \Gamma(\check{T})$ is an isomorphism, because the translations of the marked point are isotopic to the identity. We have an exact sequence

$$1 \longrightarrow \langle \delta_{\partial T} \rangle \longrightarrow \Gamma(\mathbf{T}) \longrightarrow \Gamma(\check{\mathbf{T}}) \longrightarrow 1 \,,$$

which exhibits the mapping class group of $\mathbf{T}$ as a central extension of $\mathrm{SL}(2,\mathbb{Z})$. More precisely it can be presented as

$$\Gamma(\mathbf{T}) \;\cong\; \left\langle s, \delta_\alpha \;\middle|\; s^2 = (s\delta_\alpha)^3 \right\rangle .$$

Here, s^2 is the half-Dehn twist along ∂T, and its square is therefore non-trivial in $\Gamma(\mathbf{T})$.

$\Gamma(\mathbf{T})$ acts transitively on $\hat{S}^\circ_T$, and the stabiliser of the particular element α (forgetting about its orientation) is generated by the Dehn twist along α, the Dehn twist along the boundary, and the element s^2 which reverses the orientation of (α, β). Therefore

Lemma 4.4

$$\begin{aligned} \hat{S}^\circ_T \;&\cong\; \Gamma(\mathbf{T})/\langle \delta_\alpha, \delta_{\partial T}, s^2 \rangle \;\cong\; \mathrm{SL}(2,\mathbb{Z})/\langle \delta_\alpha, s^2 \rangle \;\cong\; \mathrm{PSL}(2,\mathbb{Z})/\langle \delta_\alpha \rangle \,, \\ \hat{S}^{\circ,\mathrm{or}}_T \;&\cong\; \Gamma(\mathbf{T})/\langle \delta_\alpha, \delta_{\partial T} \rangle \;\cong\; \mathrm{SL}(2,\mathbb{Z})/\langle \delta_\alpha \rangle \,. \end{aligned}$$

■

Let $p, q \in \mathbb{Z}$ such that $\gcd(p,q) = 1$. There exists $p', q' \in \mathbb{Z}$ such that $pq' - qp' = 1$. If $p'', q'' \in \mathbb{Z}$ is another such ordered pair, there exists $r \in \mathbb{Z}$ such that $p'' = p' + rp$ and $q'' = q' + rq$. Therefore, any element x of $\mathrm{SL}(2,\mathbb{Z})/\langle \delta_\alpha \rangle$ is characterised by the ordered pair (p,q) as above, which corresponds to the oriented simple closed curve on $\check{T}$ which is homologous to $p\alpha + q\beta$. Likewise, $\mathrm{PSL}(2,\mathbb{Z})/\langle \delta_\alpha$ is in bijection with the set of ordered pairs $(p,q) \in \mathbb{Z}$ up to a global sign and such that $\gcd(p,q) = 1$. In other words

$$\hat{S}^\circ_T \;\cong\; \mathbb{Q} \cup \{\infty\}, \qquad \hat{S}^{\circ,\mathrm{or}}_T \;\cong\; (\mathbb{Q} \cup \{\infty\}) \times \{\pm 1\} \,.$$

The continuous fraction expansion of p/q give a way to represent a (canonical) $\mathrm{(P)SL}(2,\mathbb{Z})$ matrix in this coset as a product of generators δ_α and δ_β, which is also described by paths in the Farey graph. To summarise informally, the sums by which one can induce a GR amplitude for tori with one boundary are naturally sums over vertices of Farey graphs.

4.3 GR amplitude for the sphere with four boundaries

Let $\mathbf{\Sigma}_{0,4}$ be an object in Bord_1 which is a sphere with four boundaries. Any non-boundary parallel simple closed curve γ cuts $\mathbf{\Sigma}_{0,4}$ into two pairs of pants. We denote $\mathbf{P}_\gamma$ the one having $\partial_- \Sigma_{0,4}$ as one of its boundary components, $b_\gamma = \partial P_\gamma \cap \partial_+ \Sigma 0, 4$, and $\mathbf{P}'_\gamma$ the second pair of pants. Conversely, any homotopy class of embedded pair of pants bounding $\partial_- \Sigma_{0,4}$ arise in this way. Therefore, the GR amplitude for $\mathbf{\Sigma}_{0,4}$ is

$$\Omega_{\mathbf{\Sigma}_{0,4}} = \sum_{\gamma \in S^\circ_{\Sigma_{0,4}}} \mathbf{\Theta}_\gamma \big(B^{b_\gamma}_{\mathbf{P}_\gamma}, A_{\mathbf{P}'_\gamma} \big) \,.$$

We shall understand the set of simple closed curves on $\Sigma_{0,4}$, which in fact reduces to the situation of a torus with one boundary already discussed in the previous paragraph. We remark that we do not need to consider oriented curves here, as γ receives a canonical orientation by declaring that it leaves $\mathbf{P}_\gamma$ (determined by $\partial_-\Sigma$) on its left.

Let $\check{\mathbf{\Sigma}}_{0,4}$ the sphere marked with four points, obtained by glueing disks on each boundary component of $\mathbf{\Sigma}_{0,4}$, forgetting the marked point on each, and adding one marked point in the interior of each disk. Let $f : \check{\Sigma}_{1,1} \to \check{\Sigma}_{0,4}$ be a double branched cover of $\check{\Sigma}_{0,4}$ ramified at the four marked points. After applying a diffeomorphism in the source and in the target, we can assume that

$$\check{\Sigma}_{1,1} \simeq \mathbb{C}/\big(\mathbb{Z} \oplus \mathrm{i}\mathbb{Z}\big), \qquad f(z) = z \bmod \big(\tfrac{1}{2}\mathbb{Z} \oplus \tfrac{\mathrm{i}}{2}\mathbb{Z}\big), \tag{21}$$

see Figure 7.

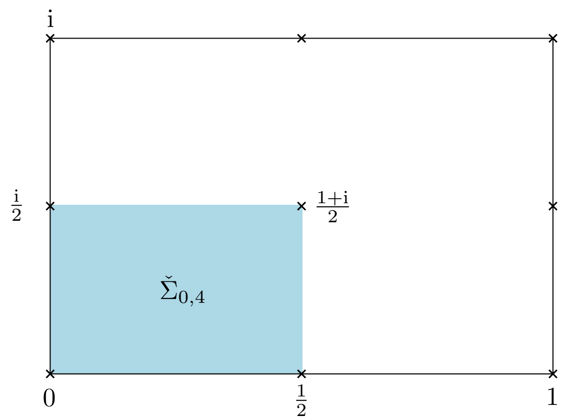

Figure 7: The double branched cover $\check{\Sigma}_{1,1}$ of $\check{\Sigma}_{0,4}$.

Lemma 4.5 *[4, Corollary 3.3] The composition of f^* with the projection $\Gamma(\check{\Sigma}_{0,4}) \cong \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{Z})$ induces an exact sequence*

$$1 \longrightarrow (\mathbb{Z}/2\mathbb{Z})^2 \longrightarrow \Gamma(\check{\mathbf{\Sigma}}_{0,4}) \longrightarrow \mathrm{PSL}(2,\mathbb{Z}) \longrightarrow 1\,,$$

where $(\mathbb{Z}/2\mathbb{Z})^2$ is the group of translations by vectors in the lattice $\frac{1}{2}\mathbb{Z}\mathrm{oplus}\frac{\mathrm{i}}{2}\mathbb{Z}$.

Proposition 4.6 *We have a bijection*

$$\hat{S}^{\circ}_{\Sigma_{0,4}} \cong \mathrm{SL}(2,\mathbb{Z})/\langle\delta_\alpha\rangle \;\cong\; (\mathbb{Q} \cup \{\infty\}) \times \{\pm 1\}\,.$$

Proof. Let γ_0 be a non-boundary parallel simple closed curve in $\Sigma_{0,4} \subset \check{\Sigma}_{0,4}$. We have

$$\hat{S}^{\circ}_{\Sigma_{0,4}} \cong \Gamma(\mathbf{\Sigma}_{0,4})/(\mathrm{Stab}\,\gamma_0) \;\cong\; \Gamma(\mathbf{\Sigma}_{0,4})/(\mathrm{Stab}\,\alpha)\,,$$

where we can choose the oriented curve α in $\check{\Sigma}_{1,1}$ to be $[0,1]$ in the model (21). As the translations by vectors in the lattice $\frac{1}{2}\mathbb{Z} \subset (\mathbb{Z}/2\mathbb{Z})^2$ stabilise α, we deduce from the exact sequence of Lemma 4.5 that

$$\hat{S}^{\circ}_{\Sigma_{0,4}} \;\cong\; \mathrm{PSL}(2,\mathbb{Z})/\langle\delta_\alpha\rangle \times \{\pm 1\} \;\cong\; \mathrm{SL}(2,\mathbb{Z})/\langle\delta_\alpha\rangle\,,$$

which according to the discussion in Section 4.2 is in bijection with $(\mathbb{Q} \cup \{\infty\}) \times \{\pm 1\}$. ∎

5 Symmetric GR

In the recursive definition (7), the single boundary component in $\partial_-\Sigma$ plays a special role, and this propagates at each step of the recursion. This is why we have formulated the construction with $\mathrm{Bord}_1^\bullet$. We now describe a variant where all boundary components play the same role, *i.e.* with $\mathrm{Bord}_s^\bullet$.

In this section, it is assumed that all surfaces are objects of $\mathrm{Bord}_s^\bullet$, unless stated otherwise. We make the preliminary remark that, to any object $\mathbf{\Sigma}$ in $\mathrm{Bord}_s^\bullet$ together with a choice of $b_1 \in \pi_0(\partial\Sigma)$, we can associate an object $\mathbf{\Sigma}^{b_1}$ in $\mathrm{Bord}_1^\bullet$, by reversing the orientation of b_1. If Σ is stable, the set of homotopy classes $\mathfrak{C}(\mathbf{\Sigma}^{b_1})$ required for the excision in Section 2.5 for the object $\mathbf{\Sigma}^{b_1}$ in $\mathrm{Bord}_1^\bullet$, is here denoted $\mathfrak{C}_{b_1}(\mathbf{\Sigma})$. As there is no ambiguity, we also denote $\mathfrak{C}_{b,b'}(\mathbf{\Sigma}) \coloneqq \mathfrak{C}_{b,b'}(\mathbf{\Sigma}^b)$.

5.1 Initial data

Take a target theory $\mathbf{E} : \mathrm{Bord}_s^\bullet \to \mathcal{C}$. We have to modify slightly the notion of admissible initial data.

Definition 5.1 *Symmetric initial data are functorial assignments*

- *of* $A_{\mathbf{P}} \in E(\mathbf{P})$ *for any pair of pants* $\mathbf{P}$,
- *of* $B_{\mathbf{P}}^{b_1,b_2} \in E(\mathbf{P})$ *for any pair of pants* $\mathbf{P}$ *together with a choice of an ordered pair* $b_1, b_2 \in \pi_0(\partial P)$,
- *of* $C_{\mathbf{P}}^{b_1} \in E(\mathbf{P})$ *for any pair of pants* $\mathbf{P}$ *together with a choice of* $b_1 \in \pi_0(\partial P)$,
- *of* $D_{\mathbf{T}} \in E(\mathbf{T})$ *for any torus* $\mathbf{T}$ *with one boundary component,*

satisfying the four axioms below with respect to glueing morphisms. The notion of admissibility is the same as in Definition 3.5.

If (γ, γ') is an ordered pair of boundary components, we denote by $\sigma_{\gamma,\gamma'}$ – called a braiding of γ and γ' – any mapping class which sends γ to γ'. We note that the validity of the relations below does not depend on the choice of the braidings $\sigma_{\gamma,\gamma'}$

BA relation. For any connected $\mathbf{\Sigma}$ with genus 0 and 4 ordered boundary components (b_1, b_2, b_3, b_4),

$$(\sigma_{b_1,b_2} - \mathrm{Id})\left(\sum_{c \in \mathfrak{C}_{b_1,b_2}(\mathbf{\Sigma})} (\mathrm{Id} + \sigma_{b_2,b_3} + \sigma_{b_2,b_4})\mathbf{\Theta}_c(B_{\mathbf{P}_c}^{b_1,b_2}, A_{\mathbf{\Sigma}_c})\right) = 0. \tag{22}$$

BB-CA relation. For any connected $\mathbf{\Sigma}$ with genus 0 and 4 ordered boundary components (b_1, b_2, b_3, b_4),

$$(\sigma_{b_1,b_2} - \mathrm{Id})\left(\sum_{c \in \mathfrak{C}_{b_1,b_2}(\mathbf{\Sigma})} \mathbf{\Theta}_c(B_{\mathbf{P}_c}^{b_1,b_2}, B_{\mathbf{\Sigma}_c}^{\gamma_c,b_3}) + \sum_{c \in \mathfrak{C}_{b_1,b_3}(\mathbf{\Sigma})} \mathbf{\Theta}_c(B_{\mathbf{P}_c}^{b_1,b_3}, B_{\mathbf{\Sigma}_c}^{b_2,\gamma_c}) + \sum_{c \in \mathfrak{C}_{b_1,b_1}^{b_4}(\mathbf{\Sigma})} \mathbf{\Theta}_c(C_{\mathbf{P}_c}^{b_1}, A_{\mathbf{\Sigma}_c})\right) = 0, \tag{23}$$

where $\mathfrak{C}_{b_1,b_1}^{b_4}(\mathbf{\Sigma})$ consists of the homotopy classes of simple curves c with endpoints in $\partial_{b_1}(\Sigma)$ modulo the group generated by the Dehn twist δ_{b_1} such that c is homotopic to $\tilde{c}^{-1} \cdot \partial_{b_4}(\Sigma) \cdot \tilde{c}$ for some $\tilde{c} \in \mathfrak{C}_{b_1,b_4}(\Sigma)$.

BC relation. For any connected $\mathbf{\Sigma}$ with genus 0 and 4 ordered boundary components (b_1, b_2, b_3, b_4),

$$(\sigma_{b_1,b_2} - \mathrm{Id})\left(\sum_{c \in \mathfrak{C}_{b_1,b_2}(\mathbf{\Sigma})} \mathbf{\Theta}_c(B_{\mathbf{P}_c}^{b_1,b_2}, C_{\mathbf{\Sigma}_c}^{\gamma_c}) + \sum_{c \in \mathfrak{C}_{b_1,b_1}^{b_3}(\mathbf{\Sigma})} \Big(\mathbf{\Theta}_c(C_{\mathbf{P}_c}^{b_1}, B_{\mathbf{\Sigma}_c}^{b_2,\gamma_c}) + \sum_{c \in \mathfrak{C}_{b_1,b_1}^{b_4}(\mathbf{\Sigma})} \mathbf{\Theta}_c(C_{\mathbf{P}_c}^{b_1}, B_{\mathbf{\Sigma}_c}^{b_2,\gamma_c^2})\Big)\right) = 0, \tag{24}$$

where (γ_c^1, γ_c^2) are the two connected components of the multicurve γ_c.

D relation. For any connected $\mathbf{\Sigma}$ of genus 1 with 2 ordered boundaries (b_1, b_2),

$$(\sigma_{b_1,b_2} - \mathrm{Id})\left(\sum_{c\in\mathfrak{C}_{b_1,b_2}(\mathbf{\Sigma})} \mathbf{\Theta}_c(B^{b_1,b_2}_{\mathbf{P}_c}, D_{\Sigma_c}) + \frac{1}{2}\sum_{c\in\mathfrak{C}_{b_1,b_1}(\mathbf{\Sigma})} \mathbf{\Theta}_c(C^{b_1}_{\mathbf{P}_c}, A_{\mathbf{\Sigma}_c})\right) = 0. \tag{25}$$

We note that, due to the properties of the initial data, the validity of these relations for some choice of braidings $\sigma_{b_1,b_2}, \sigma_{b_2,b_3}, \sigma_{b_3,b_4}$ imply their validities for all choice of braidings.

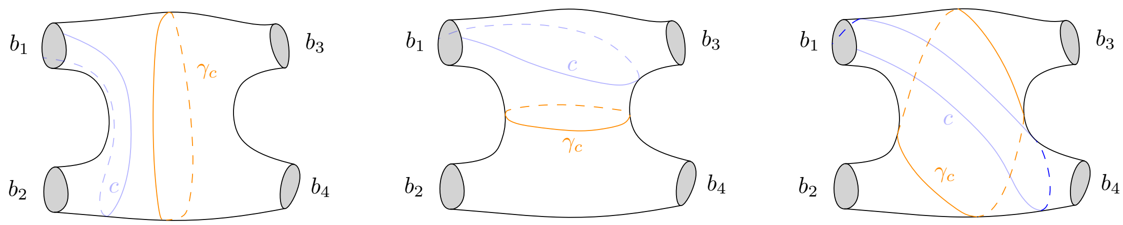

Figure 8: The three types of terms in the BB-CA relation. The same three types of terms appear in the BA and BC relations, except for a different labeling by homotopy classes c. These relations take a H-I-X form, and provide a geometric incarnation for the (purely algebraic) H-I-X relations found in quantum Airy structures and the topological recursion, see Section 7.2.

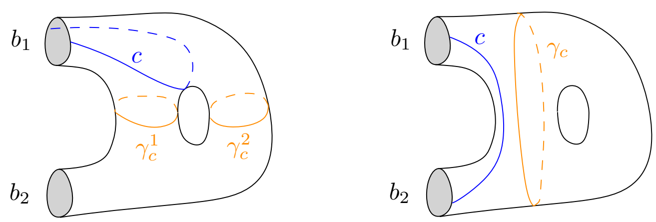

Figure 9: The two types of terms in the D-relation.

5.2 Definition

Let (A, B, C, D) be an admissible symmetric initial data for a symmetric target theory $\mathbf{E} : \mathrm{Bord}^{\bullet}_{\mathrm{s}} \to \mathcal{C}$.

Definition 5.2 *We define the symmetric GR amplitudes as follows.*

We put $\Omega_{\varnothing} := 1 \in \mathbf{E}(\varnothing)$.

For pairs of pants $\mathbf{P}$, *we put* $\Omega_{\mathbf{P}} := A_{\mathbf{P}}$.

For tori with one boundary $\mathbf{T}$, *we put* $\Omega_{\mathbf{T}} := D_{\mathbf{T}}$.

If $\mathbf{\Sigma}$ *is a connected object in* $\mathrm{Bord}^{\bullet}_{s}$ *with Euler characteristic* $\chi(\Sigma) \le -2$, *let us make a choice of a boundary component* b_1. *We now seek to inductively define* $\Omega_{\mathbf{\Sigma}}$ *using the GR formula (7)*

$$\Omega_{\mathbf{\Sigma}} := \frac{1}{2}\sum_{c\in\mathfrak{C}_{b_1,b_1}(\mathbf{\Sigma})} \mathbf{\Theta}_c(C_{\mathbf{P}_c}, \Omega_{\mathbf{\Sigma}_c}) + \sum_{b\in\pi_0(\partial_+\Sigma)}\sum_{c\in\mathfrak{C}_{b_1,b}(\mathbf{\Sigma})} \mathbf{\Theta}_c(B^b_{\mathbf{P}_c}, \Omega_{\mathbf{\Sigma}_c}). \tag{26}$$

For disconnected objects, we declare

$$\Omega_{\mathbf{\Sigma}} := \bigsqcup_{a \in \pi_0(\mathbf{\Sigma})} \Omega_{\mathbf{\Sigma}(a)}.$$

We now have to check that, step by step, this assignment does not depend on the choice of b_1.

Theorem 5.3 *The four axioms and the decay axiom imposed on admissible symmetric initial data guarantee that $\mathbf{\Sigma} \mapsto \Omega_{\mathbf{\Sigma}} \in E'(\mathbf{\Sigma}) \subset E(\mathbf{\Sigma})$ is well-defined, is independent of the choice of b_1, and is a functorial assignment.*

Proof. We denote $\Omega_{\mathbf{\Sigma}}^{b_1}$ the right-hand side of (26), to stress its potential dependence in the choice of the boundary component. The result obviously holds when $\chi(\Sigma) = -1$.

Step 1. Let us prove it when $\mathbf{\Sigma}$ has genus 0 with 4 boundary components, which we arbitrarily order (b_1, b_2, b_3, b_4). Applying (26) with the chosen boundary component b_1 yields

$$\Omega_{\mathbf{\Sigma}}^{b_1} := \sum_{c \in \mathfrak{C}_{b_1,b_2}(\mathbf{\Sigma})} \mathbf{\Theta}_c(B_{\mathbf{P}_c}^{b_1,b_2}, A_{\mathbf{\Sigma}_c}) + \sum_{c \in \mathfrak{C}_{b_1,b_3}(\mathbf{\Sigma})} \mathbf{\Theta}_c(B_{\mathbf{P}_c}^{b_1,b_3}, A_{\mathbf{\Sigma}_c}) + \sum_{c \in \mathfrak{C}_{b_1,b_4}(\mathbf{\Sigma})} \mathbf{\Theta}_c(B_{\mathbf{P}_c}^{b_1,b_4}, A_{\mathbf{\Sigma}_c}). \qquad (27)$$

By functoriality of the initial data and the glueing maps, for any morphism $\varphi : \mathbf{\Sigma} \to \mathbf{\Sigma}'$, we have $\mathbf{E}(\varphi)(\Omega_{\mathbf{\Sigma}}^{b_1}) = \Omega_{\mathbf{\Sigma}}^{\varphi(b_1)}$. For instance, the mapping class group invariance of B implies $\Omega_{\mathbf{\Sigma}}^{b_1}$ is invariant under braidings among b_2, b_3 and b_4, e.g. the first sum in (27) is obtained from the second sum by a braiding σ_{b_2,b_3} and the last sum is left invariant. The **BA** relation provides the missing invariance under braidings σ_{b_1,b_2}, *i.e.* $\sigma_{b_1,b_2}\Omega_{\mathbf{\Sigma}}^{b_1} = \Omega_{\mathbf{\Sigma}}^{b_2}$. So, $\Omega_{\mathbf{\Sigma}} := \Omega_{\mathbf{\Sigma}}^{b_1}$ is a well-defined, functorial assignment in $E(\mathbf{\Sigma})$ for surfaces $\mathbf{\Sigma}$ of this topology.

Remark. The functoriality argument was already used in the proof of Proposition 3.7, and its repetitive use in the remaining of the proof will not be mentioned anymore: it will be enough to establish the missing σ_{b_1,b_2} invariance of $\Omega_{\mathbf{\Sigma}}^{b_1}$.

Step 2. We prove the result likewise when $\mathbf{\Sigma}$ has genus 1 with 2 boundary components, which we arbitrarily order (b_1, b_2). Applying (26) with chosen boundary component b_1 yields

$$\Omega_{\mathbf{\Sigma}}^{b_1} := \sum_{c \in \mathfrak{C}_{b_1,b_1}(\mathbf{\Sigma})} \tfrac{1}{2}\mathbf{\Theta}_c(C_{\mathbf{P}_c}^{b_1}, A_{\mathbf{\Sigma}_c}) + \sum_{c \in \mathfrak{C}_{b_1,b_2}(\mathbf{\Sigma})} \mathbf{\Theta}_c(B_{\mathbf{P}_c}^{b_1,b_2}, D_{\mathbf{\Sigma}_c}),$$

and this is invariant under braidings σ_{b_1,b_2} according to relation D.

Step 3. So far we have proved the result up to $\chi(\Sigma) \geq -2$. Assume it holds for surfaces with Euler characteristic larger than a $\chi_0 \leq -2$, and consider $\mathbf{\Sigma}$ of Euler characteristic $\chi_0 - 1$.

If $\mathbf{\Sigma}$ has only 1 boundary component, then $\Omega_{\mathbf{\Sigma}}$, defined by (26), does not depend on any choice, and functoriality from the induction hypothesis, of the initial data and glueing maps imply that $\Omega_{\mathbf{\Sigma}}$ is functorial for such surfaces. If $\mathbf{\Sigma}$ has $n \geq 2$ boundary components, we order them arbitrarily $(b_1, \ldots, b_n)$ and we shall prove invariance of $\Omega_{\mathbf{\Sigma}}^{b_1}$ under the braiding σ_{b_1,b_2}. Let us compute $\Omega_{\mathbf{\Sigma}}^{b_1}$ from (26), *i.e.* excise the pair of pants $\mathbf{P}_c$ specified by all possible $c \in \mathfrak{C}_{b_1}(\mathbf{\Sigma})$. We want to replace the contribution of some of the connected components of $\mathbf{\Sigma}_c$ with the GR formula unambiguously defined from the previous induction steps. We distinguish several cases in doing so.

- When $c \in \mathfrak{C}_{b_1,b_1}(\mathbf{\Sigma})$, (γ_c^1, γ_c^2) are the connected components of the multicurve γ_c along which we excised – they are ordered by our definition of excision in § 2.5. We denote by S_c the connected component of Σ_c, which contains b_2 and S_c' the union of all other connected components of Σ_c. If S_c is a pair of pants bounded by γ_c^k for some $k \in \{1,2\}$, b_2 and another b_i for $i \geq 3$

we replace $\Omega_{\mathbf{S}_c} = A_{\mathbf{S}_c}$, and leave $\Omega_{\mathbf{S}'_c}$ as it is. We denote by $\mathfrak{C}^{b_i,k}_{b_1,b_1}(\mathbf{\Sigma})$ the set of c's such that $\pi_0(\partial S_c) = \{\gamma_c^k, b_2, b_i\}$ and S_c is a pair of pants. We observe that the contributions for $k = 1$ and 2 from $\mathfrak{C}^{b_i,k}_{b_1,b_1}(\mathbf{\Sigma})$ are equal, since $C^{b_1}_{P_c}$ is invariant under braidings of its two last boundaries. The subset $\mathfrak{C}^{(0)}_{b_1,b_1}(\mathbf{\Sigma}) \subset \mathfrak{C}_{b_1,b_1}(\mathbf{\Sigma})$ will consist of those c for which S_c is not a pair of pants. For these we replace $\Omega_{\mathbf{S}_c}$ using GR formula (26) with b_2 the chosen boundary, *i.e.* excise $\mathbf{P}_{c'}$ from $\mathbf{\Sigma}_c$ in all possible ways specified by $c' \in \mathfrak{C}_{b_2}(\mathbf{\Sigma}_c)$. If $c' \in \mathfrak{C}_{b_2,b_2}(\mathbf{\Sigma}_c)$ it yields a contribution of $C^{b_2}_{\mathbf{P}_{c'}}$ and if $c' \in \mathfrak{C}_{b_2,b_j}(\mathbf{\Sigma}_c)$, for b_j a boundary component of Σ_c, we get $B^{b_2,b_j}_{\mathbf{P}_{c'}}$.

- When $c \in \mathfrak{C}_{b_1,b_2}(\mathbf{\Sigma})$, we replace $\Omega_{\mathbf{\Sigma}_c}$ using the GR formula (26) with γ_c as the chosen boundary component. We are therefore excising a second time with all possible $c' \in \mathfrak{C}_{\gamma_c}(\mathbf{\Sigma}_c)$, and denote $\mathbf{\Sigma}_{c,c'} = \mathbf{\Sigma}_c - \mathbf{P}_{c'}$ the excised surface. If $c' \in \mathfrak{C}_{\gamma_c,\gamma_c}(\mathbf{\Sigma}_c)$, we get a contribution of $C^{\gamma_c}_{\mathbf{P}_{c'}}$, while if $c' \in \mathfrak{C}_{\gamma_c,b_i}(\mathbf{\Sigma}_c)$ for some $i \geq 3$ we get a contribution of $B^{\gamma_c,b_i}_{\mathbf{P}_{c'}}$.
- When $c \in \mathfrak{C}_{b_1,b_i}(\mathbf{\Sigma})$ for $i \geq 3$, we replace $\Omega_{\mathbf{\Sigma}_c}$ using GR formula (26) with b_2 the chosen boundary component, *i.e.* perform a second excision specified by $c' \in \mathfrak{C}_{b_2}(\mathbf{\Sigma}_c)$. We denote $\mathbf{\Sigma}_{c,c'} = \mathbf{\Sigma}_c - \mathbf{P}_{c'}$. If $c' \in \mathfrak{C}_{b_2,b_2}(\mathbf{\Sigma}_c)$, we denote $(\gamma^1_{c'}, \gamma^2_{c'})$ the ordered connected components of the multicurve $\gamma_{c'}$, and we get a contribution of $C^{b_2}_{\mathbf{P}_{c'}}$. For $c' \in \mathfrak{C}_{b_2,\gamma_c}(\mathbf{\Sigma}_c)$, we get $B^{b_2,\gamma_c}_{\mathbf{P}_{c'}}$. In the remaining cases, $c' \in \mathfrak{C}_{b_2,b_j}(\mathbf{\Sigma}_c)$ for some $j \geq 3$ with $j \neq i$, we get a contribution of $B^{b_2,b_j}_{\mathbf{P}_{c'}}$.

Keeping in mind the constraint $\chi(\Sigma) < -2$, the list above exhausts all possible situations, in particular this process will not pull out contributions of a D. We also stress that $\Sigma_{c,c'}$ may be empty in some situations, in which case we use the convention $\Omega_\varnothing = 1 \in \mathbb{K}$ and the union axiom of § 3.1 for the emptyset. The outcome is

$$
\begin{aligned}
&\Omega^{b_1}_{\mathbf{\Sigma}} \qquad (28)\\
&= \sum_{i=3}^{n} \sum_{c \in \mathfrak{C}^{b_i,1}_{b_1,b_1}(\mathbf{\Sigma})} \mathbf{\Theta}_c(C^{b_1}_{\mathbf{P}_c}, A_{\mathbf{S}_c}, \Omega_{\mathbf{S}'_c}) \\
&+ \sum_{c \in \mathfrak{C}^{(0)}_{b_1,b_1}(\mathbf{\Sigma})} \sum_{c' \in \mathfrak{C}_{b_2,b_2}(\mathbf{\Sigma}_c)} \tfrac{1}{4}\mathbf{\Theta}_c\big(C^{b_1}_{\mathbf{P}_c}, \mathbf{\Theta}_{c'}(C^{b_2}_{\mathbf{P}_{c'}}, \Omega_{\mathbf{\Sigma}_{c,c'}})\big) + \sum_{i=3}^{n} \sum_{c \in \mathfrak{C}^{(0)}_{b_1,b_1}(\mathbf{\Sigma})} \sum_{c' \in \mathfrak{C}_{b_2,b_i}(\mathbf{\Sigma}_c)} \tfrac{1}{2}\mathbf{\Theta}_c\big(C^{b_1}_{\mathbf{P}_c}, \mathbf{\Theta}_{c'}(B^{b_2,b_i}_{\mathbf{P}_{c'}}, \Omega_{\mathbf{\Sigma}_{c,c'}})\big) \\
&+ \sum_{c \in \mathfrak{C}^{(0)}_{b_1,b_1}(\mathbf{\Sigma})} \Bigg\{ \sum_{c' \in \mathfrak{C}_{b_2,\gamma^1_c}(\mathbf{\Sigma}_c)} \tfrac{1}{2}\mathbf{\Theta}_c\big(C^{b_1}_{\mathbf{P}_c}, \mathbf{\Theta}_{c'}(B^{b_2,\gamma^1_c}_{\mathbf{P}_{c'}}, \Omega_{\mathbf{\Sigma}_{c,c'}})\big) + \sum_{c' \in \mathfrak{C}_{b_2,\gamma^2_c}(\mathbf{\Sigma}_c)} \tfrac{1}{2}\mathbf{\Theta}_c\big(C^{b_1}_{\mathbf{P}_c}, \mathbf{\Theta}_{c'}(B^{b_2,\gamma^2_c}_{\mathbf{P}_{c'}}, \Omega_{\mathbf{\Sigma}_{c,c'}})\big)\Bigg\} \\
&+ \sum_{c \in \mathfrak{C}_{b_1,b_2}(\mathbf{\Sigma})} \Bigg\{ \sum_{c' \in \mathfrak{C}_{\gamma_c,\gamma_c}(\mathbf{\Sigma}_c)} \tfrac{1}{2}\mathbf{\Theta}_c\big(B^{b_1,b_2}_{\mathbf{P}_c}, \mathbf{\Theta}_{c'}(C^{\gamma_c}_{\mathbf{P}_{c'}}, \Omega_{\mathbf{\Sigma}_{c,c'}})\big) + \sum_{i=3}^{n} \sum_{c' \in \mathfrak{C}_{\gamma_c,b_i}(\mathbf{\Sigma}_c)} \Theta_c\big(B^{b_1,b_2}_{\mathbf{P}_c}, \mathbf{\Theta}_{c'}(B^{\gamma_c,b_i}_{\mathbf{P}_{c'}}, \Omega_{\mathbf{\Sigma}_{c,c'}})\big)\Bigg\} \\
&+ \sum_{i=3}^{n} \sum_{c \in \mathfrak{C}_{b_1,b_i}(\mathbf{\Sigma})} \Bigg\{ \sum_{c' \in \mathfrak{C}_{b_2,b_2}(\mathbf{\Sigma}_c)} \tfrac{1}{2}\mathbf{\Theta}_c\big(B^{b_1,b_i}_{\mathbf{P}_c}, \mathbf{\Theta}_{c'}(C^{b_2}_{\mathbf{P}_{c'}}, \Omega_{\mathbf{\Sigma}_{c,c'}})\big) + \sum_{c' \in \mathfrak{C}_{b_2,\gamma_c}(\mathbf{\Sigma}_c)} \mathbf{\Theta}_c\big(B^{b_1,b_i}_{\mathbf{P}_c}, \mathbf{\Theta}_{c'}(B^{b_2,\gamma_c}_{\mathbf{P}_{c'}}, \Omega_{\mathbf{\Sigma}_{c,c'}})\big) \\
&\qquad + \sum_{\substack{j \geq 3 \\ j \neq i}} \sum_{c' \in \mathfrak{C}_{b_2,b_j}(\mathbf{\Sigma}_c)} \mathbf{\Theta}_c\big(B^{b_1,b_i}_{\mathbf{P}_c}, \mathbf{\Theta}_{c'}(B^{b_2,b_j}_{\mathbf{P}_{c'}}, \Omega_{\mathbf{\Sigma}_{c,c'}})\big)\Bigg\}.
\end{aligned}
$$

We have ten multisums on the right, which we will denote $\mathscr{S}_i$, $i = 1, \ldots 10$ – illustrated below.

First we observe that $\mathscr{S}_2$ is a sum over the set of $(c, c') \in \mathfrak{C}_{b_1,b_1}(\mathbf{\Sigma}) \times \mathfrak{C}_{b_2,b_2}(\mathbf{\Sigma})$, such that there exist non-intersecting representatives and cutting along $\gamma_c \cup \gamma_{c'}$ will not produce a component which is a pair of pants with b_2 and some other b_i (b_1 and some other b_i) in its boundary. This sum is manifestly invariant under σ_{b_1,b_2}. The same kind of argument also applies to $\mathscr{S}_{10}$.

Inspection of $\mathscr{S}_3 + \mathscr{S}_8$ reveals that the sum of these two terms must be invariant under σ_{b_1,b_2}, since we can use composition of glueings on one multicurve at a time.

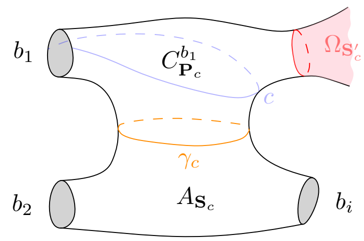

Figure 10: $\mathcal{S}_1$

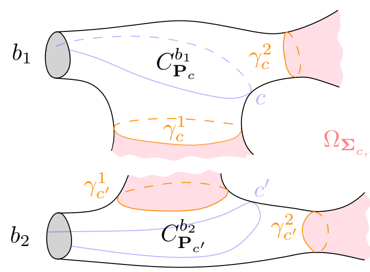

Figure 11: $\mathcal{S}_2$

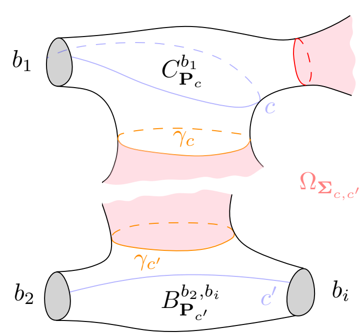

Figure 12: $\mathcal{S}_3$

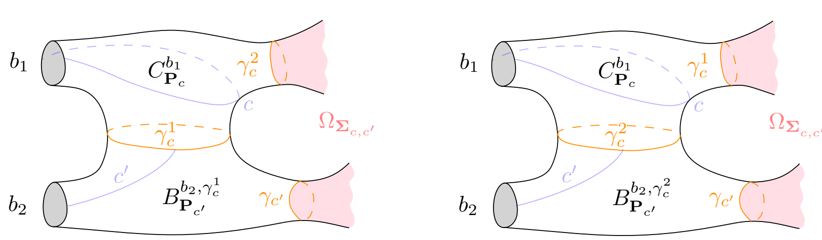

Figure 13: $\mathcal{S}_4 + \mathcal{S}_5$

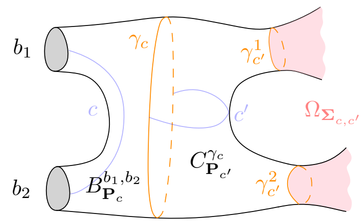

Figure 14: $\mathcal{S}_6$

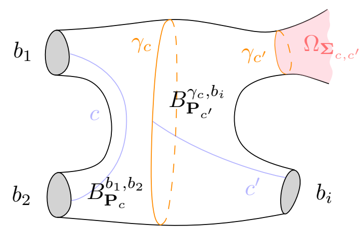

Figure 15: $\mathcal{S}_7$

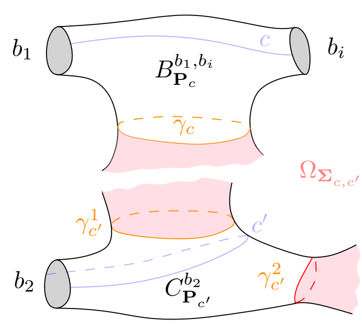

Figure 16: $\mathcal{S}_8$. γ_c may or may not coincide with $\gamma^k_{c'}$ for some $k \in \{1,2\}$.

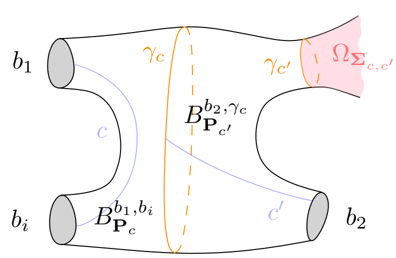

Figure 17: $\mathcal{S}_9$.

We now consider $\mathscr{S}_9 + \mathscr{S}_7 + \mathscr{S}_1$. First we invoke composition of glueing for the summands in $\mathscr{S}_9$ and $\mathscr{S}_7$, so as to reduce each summand to one glueing along one multicurve. If we now compare the summands of $\mathscr{S}_9$, $\mathscr{S}_7$ and $\mathscr{S}_1$ with the three terms in the BB-CA relation, we find they match. We now reorganise the three multisums of $\mathscr{S}_9$, $\mathscr{S}_7$ and $\mathscr{S}_1$, into a sum over all possible isotopy classes of embeddings of a surface S of genus 0 and 4 boundary components, where 3 of its boundary components goes to b_1, b_2 and b_i, for some fixed $i \geq 3$, followed by a sum as in the BB-CA relation locally in each

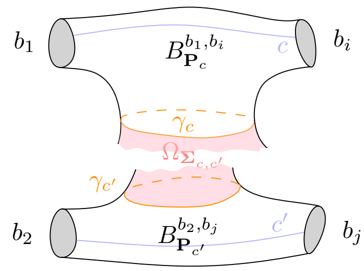

Figure 18: $\mathcal{S}_{10}$.

of these embedded surfaces S. For each embedding of S we can now invoke the BB-CA relation to conclude that all together $\mathscr{S}_9 + \mathscr{S}_7 + \mathscr{S}_1$ is invariant under σ_{b_1,b_2}.

Finally, we consider $\mathscr{S}_4 + \mathscr{S}_5 + \mathscr{S}_6$. We reorganise these three sums to first be a sum over all isotopy classes of embeddings of a surface S of genus zero and four boundary components, where two of its boundary components goes to b_1, b_2 and the other two are mapped to the interior of $\mathbf{\Sigma}$, followed by a sum as in the BC relation in each of these embedded surfaces S. Examining the summands of $\mathscr{S}_4, \mathscr{S}_5$ and $\mathscr{S}_6$ and using associativity of glueings, we observe that for each embedding of S, we can now invoke the BC relation to conclude all together that $\mathscr{S}_4 + \mathscr{S}_5 + \mathscr{S}_6$ is invariant under σ_{b_1,b_2}. ∎

Remark 5.4 *It is easy to see that this proof is still valid if the BB-CA relation is replaced by the relation*

$$(\sigma_{b_1,b_2} - \mathrm{Id})\left(\sum_{c \in \mathfrak{C}_{b_1,b_2}(\mathbf{\Sigma})} \mathbf{\Theta}_c(B^{b_1,b_2}_{\mathbf{P}_c}, B^{b_3,\gamma_c}_{\mathbf{\Sigma}_c}) + \sum_{c \in \mathfrak{C}_{b_1,b_3}(\mathbf{\Sigma})} \mathbf{\Theta}_c(B^{b_1,b_3}_{\mathbf{P}_c}, B^{b_2,\gamma_c}_{\mathbf{\Sigma}_c}) + \sum_{c \in \mathfrak{C}^{b_4}_{b_1,b_1}(\Sigma)} \mathbf{\Theta}_c(C^{b_1}_{\mathbf{P}_c}, A_{\mathbf{\Sigma}_c})\right) = 0 \tag{29}$$

obtained by replacing $B^{\gamma_c,b_3}_{\mathbf{\Sigma}_c}$ by $B^{b_3,\gamma_c}_{\mathbf{\Sigma}_c}$ in the BB-CA relation. This correspond to choosing b_3 instead of γ_c as the reference boundary in the step 3 of the proof when $c \in \mathfrak{C}_{b_1,b_2}(\mathbf{\Sigma})$. By the induction hypothesis, this choice does not affect the result.

5.3 Main properties

There is a forgetful functor $\mathrm{Bord}^{\bullet}_1 \to \mathrm{Bord}^{\bullet}_s$ which reverses the orientation of the $-$ boundary component, and it is compatible with union and glueing. If $\mathbf{E}_{\mathrm{s}} : \mathrm{Bord}^{\bullet}_{\mathrm{s}} \to \mathcal{C}$ is a target theory, so is the composition of this forgetful functor with $\mathbf{E}_{\mathrm{s}}$, which we denote $\mathbf{E}$. If (A,B,C,D) is an $\mathbf{E}_{\mathrm{s}}$-valued admissible symmetric initial data and $\Omega^{\mathrm{s}}_{\mathbf{\Sigma}}$ the outcome of symmetric GR, then (A,B,C,D) can be considered as an $\mathbf{E}$-valued convergent initial data in the obvious way. And, for this convergent initial data, GR generates $\Omega_{\mathbf{\Sigma}} = \Omega^{\mathrm{s}}_{\mathbf{\Sigma}}$, as the defining formula (7) is identical to the symmetric version (26).

This little argument shows that symmetric GR is a particular case of GR where the initial data satisfies relations. Therefore, all the properties listed in § 3.4-3.5 also hold for symmetric GR.

6 Strict GR

Suppose we have chosen for each stable (g,n), a single reference object $\mathbf{\Sigma}_{g,n}$ in $\mathrm{Bord}^{\bullet}_1$ or $\mathrm{Bord}^{\bullet}_{\mathrm{s}}$ of genus g with n boundary components. In the geometric recursion, the functoriality of $\mathbf{\Sigma} \mapsto \Omega_{\mathbf{\Sigma}}$ allows a non-ambiguous definition of

$$\varpi_{g,n} := \Omega_{\mathbf{\Sigma}_{g,n}}$$

by induction on $2g-2+n>0$. Namely, it is possible to work (using the remark above the Convergence axiom) solely with reference surfaces for each g and n all the way through the recursion. Passing from Ω_Σ to $\varpi_{g,n}$, we have lost the memory of the structure of $\mathrm{Bord}_1^\bullet$ or $\mathrm{Bord}_{\mathrm{s}}^\bullet$, and therefore a lot of topological (and interesting) information and naturality of the construction.

In this section, we want to formalise the type of recursion satisfied by $(\varpi_{g,n})_{g,n}$, called *strict geometric recursion*, which is a kind of generalisation of the topological recursion of [26, 37, 2], and is only based on finitely many glueing maps (Section 6.1). This construction contains less information than GR, but it can also be induced from the richer context of GR when the glueing maps are assembled from a converging series of homotopy class-dependent glueing maps as in (13) (see Section 6.3).

6.1 Definition

We form a category $\underline{\mathrm{Bord}}_1^\bullet$ which is a kind of strictification of $\mathrm{Bord}_1^\bullet$. Its objects are finite (possibly empty) sequences $(g_i,n_i)_{i\in I}$ and $2-2g_i-n_i<0$ for all $i\in I$, and concatenation $\amalg$ of such sequences gives a monoidal structure. There are no morphisms between distinct objects, and the automorphisms of $(g_i,n_i)_{i\in I}$ is the product of the permutation groups $\prod_{i\in I}\mathfrak{S}_{[\![2,n_i]\!]}$.

Definition 6.1 *A* strict target theory *is a functor* $\underline{E}:\underline{\mathrm{Bord}}_1^\bullet\to\mathcal{C}$ *together with extra data satisfying the union and glueing axioms below.*

Union axiom. For any objects s and s', we ask for the data of a continuous bilinear map

$$\sqcup:\underline{E}(s)\times\underline{E}(s)\to\underline{E}(s\amalg s'),$$

compatible with associativity of cartesian products. We require that $\underline{E}(\varnothing):=\mathbb{K}$, and the union map $\sqcup:\underline{E}(\varnothing)\times\underline{E}(s)\to\underline{E}(s)$ is specified by $1\sqcup v=v$.

We need a few more notations before presenting the glueing axiom. If $g\geq 0$ and $n\geq 1$ are such that $2g-2+n\geq 2$, we introduce the set $\mathfrak{K}(g,n)$ of $(0,3)$-excisions. It collects the following objects

- **I** $(g-1,n+1)$.
- **I'** for each $j\in[\![2,n]\!]$, a copy of the object $(g,n-1)$ which we denote $(g,n-1)_j$.
- **II** for each ordered partition $J\cup J'=[\![2,n]\!]$ and ordered pair (h,h') such that $h+h'=g$ such that $2-2h-|J|<0$ and $2-2h'-|J'|<0$, a copy of the object $((h,1+|J|),(h',1+|J'|))$, which we denote $((h,1+J),(h',1+J'))$.

To any $\sigma\in\mathfrak{S}_{[\![2,n]\!]}$ and $\kappa\in\mathfrak{K}(g,n)$, we assign a morphism $\tilde{\sigma}:\kappa\to\kappa^\sigma$ between κ and an object $\kappa^\sigma\in\mathfrak{K}(g,n)$ in the following way.

- **I** Let $\varphi:[\![3,n+1]\!]\to[\![2,n]\!]$ defined by $\varphi(k)=k-1$. We take $(g-1,n+1)^\sigma:=(g-1,n+1)$, and $\tilde{\sigma}$ is the morphism $\varphi\circ\sigma\circ\varphi^{-1}$.
- **I'** Let φ be the unique strictly increasing map from $[\![2,n-1]\!]$ to $[\![2,n]\!]\setminus\{j\}$. $\tilde{\sigma}$ is a morphism from the j-th copy of $(g,n-1)$ to the $\sigma(j)$-th copy, and seen as an automorphism of the single object $(g,n-1)$ in $\underline{\mathrm{Bord}}_1^\bullet$, it is $\varphi\circ\sigma\circ\varphi^{-1}$.
- **II** If K is an ordered set, let φ_K the unique strictly increasing map from K to $[\![2,1+|K|]\!]$. We take $((h,1+J),(h',1+J'))^\sigma:=((h,1+\sigma(J)),(h',1+\sigma(J')))$ and $\tilde{\sigma}$ considered as an automorphism of the single object $((h,1+|J|),(h,1+|J'|))$, is $(\varphi_{\sigma(J)}\circ\sigma\circ\varphi_J^{-1})\amalg(\varphi_{\sigma(J')}\circ\sigma\circ\varphi_{J'}^{-1})$.

If $\kappa = (g_i, n_i)_{i \in I}$ is an element of $\mathfrak{K}(g,n)$, we denote $\underline{E}^{\pi}(\kappa) := \prod_{i \in I} \underline{E}(g_i, n_i)$. From the previous construction, each $\sigma \in \mathfrak{S}_{[\![2,n]\!]}$ induces morphisms $\underline{E}_{\kappa,\sigma} : \underline{E}^{\pi}(\kappa) \to \underline{E}(\kappa^{\sigma})$ for any $\kappa \in \mathfrak{K}(g,n)$.

Glueing axiom. For any (g,n) such that $2g-2+n \geq 2$ and any $\kappa \in \mathfrak{K}(g,n)$, we ask for the data of continuous (multi)linear glueing map $\mathbf{\Theta}_{\kappa} : \underline{E}(0,3) \times \underline{E}^{\pi}(\kappa) \to \underline{E}(g,n)$ which are compatible with the action of $\mathfrak{S}_{[\![2,n]\!]}$ on the second factor of the source, and on $\underline{E}(g,n)$.

Definition 6.2 *A strict initial data is a quadruple (A,B,C,D) such that*

$$A, C \in \underline{E}(0,3)^{\mathfrak{S}_2}, \qquad B^2 \in \underline{E}(0,3), \qquad D \in \underline{E}(1,1).$$

Let (A,B,C,D) be a strict initial data for a strict target theory $\underline{E} : \underline{\mathrm{Bord}}_1^{\bullet} \to \mathfrak{C}$.

Definition 6.3 *We define the strict GR amplitudes as follows.*

We put $\varpi_{\varnothing} := 1 \in \mathbb{K}$, $\varpi_{0,3} := A$ and $\varpi_{1,1} := D$.

If $2g-2+n \geq 2$, we seek to define inductively

$$\begin{aligned} \varpi_{g,n} \quad := \quad & \sum_{j=2}^{n} \mathbf{\Theta}_{(g,n-1)_j}(B^j, \varpi_{g,n-1}) \\ & + \tfrac{1}{2}\mathbf{\Theta}_{(g-1,n+1)}(C, \varpi_{g-1,n+1}) + \tfrac{1}{2} \sum_{\substack{h+h'=g \\ J \cup J' = [\![2,n]\!]}}^{\text{stable}} \mathbf{\Theta}_{((h,J),(h',J'))}(C, \varpi_{h,|J|}, \varpi_{h',|J'|}), \end{aligned} \tag{30}$$

where the middle sum is restricted to terms which are not involving the (non-defined) $(0,1)$ or $(0,2)$.

For disconnected objects, we put $\varpi_{(g_i,n_i)_{i \in I}} := \bigsqcup_{i \in I} \varpi_{g_i,n_i}$.

Proposition 6.4 *$(g,n) \mapsto \varpi_{g,n} \in \underline{E}(g,n)$ is a well-defined, functorial assignment element. In particular, $\varpi_{g,n}$ is invariant under the action of $\mathrm{Aut}(g,n) = \mathfrak{S}_{n-1}$.*

Proof. The proof proceeds by induction, and is very easy. Indeed, the only terms in (30) which may not already be symmetric thanks to the induction hypothesis is $\mathbf{\Theta}_{(g,n-1)_j}(B^j, \varpi_{g,n-1})$ indexed by $j \in [\![2,n]\!]$, but the required properties of the glueing maps imply that the automorphism group $\mathfrak{S}_{[\![2,n]\!]}$ of (g,n) just permutes these terms. ∎

6.2 Sums over fatgraphs

We resume the discussion of Section 3.5. Recall that any $G \in \mathbb{G}_1^{g,n}$ determines a pair of pants decomposition, with ordered pair of pants $(P_1, \ldots, P_{2g-2+n})$, as well as a type map $X : [\![1, 2g-2+n]\!] \to \{A,B,C,D\}$. In a strict target theory $\underline{E}$, we can assemble a glueing morphism

$$\mathbf{\Theta}_G : \prod_{i=1}^{3g-3+n} \underline{E}(P_i) \longrightarrow \underline{E}(g,n).$$

Unfolding the recursive definition (30) as we did in Section 3.5 (without the complications due to homotopy class considerations) yields

Proposition 6.5 *Let (A,B,C,D) be an admissible initial data for a strict target theory $\underline{E}$. For any stable (g,n), the strict GR amplitude take the form*

$$\varpi_{g,n} = \sum_{G \in \mathbb{G}_1^{g,n}} \mathbf{\Theta}_G\big((X_{P_i})_{i=1}^{3g-3+n}\big).$$

∎

6.3 From GR to strict GR

Assume E is a target theory satisfying the uniform convergence axiom. We can construct from E a strict target theory $\underline{E}$. This is done by choosing, for each (g,n), a reference surface $\mathbf{\Sigma}_{g,n}$ in $\mathrm{Bord}_1^{\bullet}$ and set

$$\underline{E}(g,n) := \mathbf{E}(\mathbf{\Sigma}_{g,n})^{\Gamma^{\partial}(\mathbf{\Sigma}_{g,n})}.$$

The action of $\Gamma(\mathbf{\Sigma}_{g,n})$ on $\mathbf{E}(\mathbf{\Sigma}_{g,n})$ factors into an action of $\mathfrak{S}_{\pi_0(\partial_+\Sigma_{g,n})}$ on $\underline{E}(g,n)$. Making the extra choice of an identification $\pi_0(\partial_+\Sigma_{g,n}) \simeq [\![2,n]\!]$, this group defines the action of $\mathfrak{S}_{[\![2,n]\!]}$ on $\underline{E}(g,n)$, hence defining a functor $\underline{E} : \underline{\mathrm{Bord}}_1^{\bullet} \to \mathfrak{C}$.

We induce union maps for $\underline{E}$ from the union maps on $\mathbf{E}$ in the obvious way. To define glueing maps for $\underline{E}$, we assemble the glueing maps for E. More precisely, if $\kappa \in \mathfrak{K}(g,n)$, let $\mathbf{\Sigma}_\kappa$ denote, among the reference surfaces we have chosen, the one with topology κ. We let $\mathfrak{C}(\mathbf{\Sigma}_{g,n},\kappa) \subset \mathfrak{C}(\mathbf{\Sigma}_{g,n})$ be the set of homotopy classes c such that there exists a bijection $p : \pi_0(\Sigma_\kappa) \to \pi_0(\Sigma - P_c)$, which respects the ordering of connected components when Σ_κ is disconnected, together with isomorphisms $\varphi_c : \mathbf{\Sigma}_{0,3} \to \mathbf{P}_c$ and $\psi_c := \big(\psi_{c,S} : S \to p(S)\big)_{S\in\pi_0(\Sigma_\kappa)}$ in $\mathrm{Bord}_1^{\bullet}$. For each $c \in \mathfrak{C}(\mathbf{\Sigma}_{g,n},\kappa)$ we pick such a (p,φ_c,ψ_c), and remark that any other choice is related to (φ_c,ψ_c) by postcomposition with pure mapping classes on the connected components of $\Sigma_{0,3} \cup \Sigma_\kappa$. Then, due to the convergence axiom and Lemma 3.7

$$\mathbf{\Theta}_\kappa := \sum_{c\in\mathfrak{C}(\Sigma_{g,n},\kappa)} \mathbf{\Theta}_c \circ \big(\mathbf{E}(\varphi_c), \mathbf{E}(\psi_{c,S})_{S\in\pi_0(\Sigma_\kappa)}\big) \tag{31}$$

is a well-defined multilinear map from $\underline{E}(\mathbf{\Sigma}_{0,3}) \times \underline{E}^{\pi}(\mathbf{\Sigma}_\kappa) \to \mathbf{E}(\mathbf{\Sigma}_{g,n})$, which does not depend on the choices of (p,φ_c,ψ_c) as above. Since we assumed the uniform convergence axiom, this is in fact a continuous map. And, as all elements in the sum (31) are related by action of pure mapping classes of $\mathbf{\Sigma}_{g,n}$, $\mathbf{\Theta}_\kappa$ actually takes values in $\underline{E}(g,n)$.

Using the short argument above Lemma 3.7 to work only with reference surfaces, we find by construction of (31)

Corollary 6.6 *Assume* $\mathbf{E}$ *is a target theory, together with an initial data* (A,B,C,D)*, and denote* Ω *be the outcome of GR. Construct a (non-canonical) strict target theory* $\underline{E}$ *as above, induce* $\underline{E}$*-valued initial data by specialising* (A,B,C) *to the reference* $\mathbf{\Sigma}_{0,3}$ *and* D *to the reference* $\mathbf{\Sigma}_{1,1}$*, and denote* ω *the outcome of strict GR. Then for stable* (g,n)*,* $\varpi_{g,n} = \Omega_{\mathbf{\Sigma}_{g,n}}$ ■

We note that the above construction is non-canonical as it involves the choice of reference surfaces $\mathbf{\Sigma}_{g,n}$. However, if the target theory $\mathbf{E}$ is such that $\Gamma(\mathbf{\Sigma})$ acts trivially $\mathbf{E}(\mathbf{\Sigma})$, the construction is independent of these choices and therefore canonical.

More generally, we have by an argument similar to Proposition 3.8

Proposition 6.7 *Let* $\mathbf{E}$ *be a target theory,* $\underline{E}$ *a strict target theory, and* $\eta : \mathbf{E} \Rightarrow \underline{E}$ *a natural transformation which is compatible with the union and glueing morphisms. If* $\mathfrak{I} = (A,B,C,D)$ *is an* $\mathbf{E}$*-valued admissible initial data, then* $\eta(\mathfrak{I})$ *is a strict initial data, and we have the relation between corresponding GR and strict GR amplitudes*

$$\omega^{\eta(\mathfrak{I})} = \eta(\Omega^{\mathfrak{I}}).$$

6.4 Strict symmetric GR

One can also formulate a symmetric version of the strict GR. The category $\underline{\mathrm{Bord}}_{\mathrm{s}}^{\bullet}$ is constructed in the same way as $\underline{\mathrm{Bord}}_{\mathrm{s}}^{\bullet}$, except that we now define automorphisms in this category by choosing braiding representatives for each permutation of $[\![1,n]\!]$, hence replace everywhere the groups $\mathfrak{S}_{[\![2,n]\!]}$

with $\mathfrak{S}_{[\![1,n]\!]} \cong \mathfrak{S}_n$. Strict symmetric target theories in this context are functors $\underline{E} : \underline{\mathrm{Bord}}^{\bullet}_{\mathrm{s}} \to \mathcal{C}$ satisfying axioms parallel to § 6.1.

Definition 6.8 *Initial data for strict symmetric GR is a quadruple* (A, B, C, D) *where*

$$A \in \underline{E}(0,3)^{\mathfrak{S}_3}, \qquad B \in \underline{E}(0,3), \qquad C \in \underline{E}(0,3)^{\mathfrak{S}_{\{2,3\}}}, \qquad D \in \underline{E}(1,1)\,.$$

We also require that the initial data satisfies the 4 relations below. The first three ones involve glueing maps in $(0,4)$*, the last one glueing maps in* $(2,1)$*, and* $\sigma_{i,j}$ *stands for the transposition of* i *and* j*.*

BA RELATION.

$$(\sigma_{1,2} - \mathrm{Id})\big(\mathbf{\Theta}_{(0,3)_2}(B,A) + \mathbf{\Theta}_{(0,3)_3}(B,A) + \mathbf{\Theta}_{(0,3)_4}(B,A)\big) = 0\,.$$

BB-AC RELATION.

$$(\sigma_{1,2} - \mathrm{Id})\big(\mathbf{\Theta}_{(0,3)_2}(B,B) + \mathbf{\Theta}_{(0,3)_3}(B,B) + \mathbf{\Theta}_{(0,3)_4}(C,A)\big) = 0\,.$$

BC RELATION.

$$(\sigma_{1,2} - \mathrm{Id})\big(\mathbf{\Theta}_{(0,3)_2}(B,C) + \mathbf{\Theta}_{(0,3)_3}(C,B) + \mathbf{\Theta}_{(0,3)_4}(C,B)\big) = 0\,.$$

D RELATION.

$$(\sigma_{1,2} - \mathrm{Id})\big(\mathbf{\Theta}_{(1,1)}(B,D) + \tfrac{1}{2}\mathbf{\Theta}_{(0,3)}(C,A)\big) = 0\,.$$

Given a strict symmetric target theory and initial data, we define $\varpi_s \in \underline{E}(s)$ for any object $s \in \underline{\mathrm{Bord}}^{\bullet}_{\mathrm{s}}$ by the same formulas as in Definition 6.3.

Proposition 6.9 *For any stable* (g,n)*,* $\varpi_{g,n}$ *is a well-defined element of* $\underline{E}(g,n)^{\mathfrak{S}_n}$*. In other words,* $s \mapsto \varpi_s$ *is a functorial assignment from* $\underline{\mathrm{Bord}}^{\bullet}_{\mathrm{s}}$*.*

Proof. The proof uses the same recollection of terms as done in the proof of functoriality of the symmetric GR in Theorem 5.3, with only (heavily simplifying) difference that we do not have to carry the sums over homotopy class: each of them is replaced by one of our $(0,3)$-glueing maps, as is already apparent in comparing the 4 relations here to the 4 relations in § 5.1. ■

As explained for GR/symmetric GR in § 5.3, strict symmetric GR inherits the properties of strict GR described in Section 6.3 in the obvious way, upon replacing everywhere $\mathfrak{S}_{[\![2,n]\!]}$ by $\mathfrak{S}_n$ allowing the permutation of all boundary components in $\Sigma_{g,n}$.

7 Relation to topological recursion

This section gives examples of strict symmetric GR which demonstrate that the topological recursion (TR) and its many variants are particular cases of the strict symmetric GR. There is nothing really new here, but we include these examples for pedagogical reasons. The most obvious class of example comes from two-dimensional topological quantum field theories (TQFTs). We then explain separately the class of examples governing the original topological recursion of [23] which is based on the geometry of complex curves, and the Kontsevich-Soibelman approach to topological recursion (KS-TR) [37, 2] based on algebraic structures called "quantum Airy structures". Although KS-TR contains the other two examples as particular cases, it is instructive to see in each different language how the target theory should be formed.

The strict GR can be described in some sense as a (possibly non-symmetric) generalisation of the topological recursion (TR) of [23, 37] in which the glueing maps are allowed to "increase complexity", aligning with the possibly increasing complexity of the spaces $\underline{E}(g,n)$ when $2g-2+n$ increases. This should be compared with Section 7.1-7.2 where they do not depend on g. Adopting this perspective, we may say in light of Corollary 6.6 that a natural transformation from a target theory to a strict target theory maps a GR to a TR. The richer structure which appears in the construction of GR due to infinite sums over homotopy class mapping class group considerations is not seen at the level of the corresponding TR, as everything has been incorporated into finitely many glueing maps.

Conversely, if we start from a strict target theory $\underline{E}$ and initial data, we will describe in § 10.3 a way to construct some target theory $\mathbf{E}$ and initial data such that GR recomputes the outcome of strict GR. This is based on the idea of fibering $\underline{E}$ over Teichmüller spaces, and on Mirzakhani's generalisation of McShane identity.

We nevertheless stress that target theories are allowed to contain much more information than their strict counterpart. Indeed, they are functors from the category $\mathrm{Bord}_1^\bullet$ which reflects (2+1)-dimensional topology (surfaces and their mapping classes) while $\underline{\mathrm{Bord}}_1^\bullet$ only reflects 2-dimensional topology and therefore has a simple combinatorial structure. The spaces $\mathbf{E}(\mathbf{\Sigma})$ in which one can choose to work often have an interesting geometric meaning, and do come naturally (functorially) with non-trivial mapping class group actions. This meaning is carried to the elements $\Omega_{\mathbf{\Sigma}}$ we produce by GR. The gain between GR and strict GR will become clear in Chapter II (and III), where we construct and use the first non-trivial examples of target theories based on the topology (the geometry) on the Teichmüller spaces, and provide examples of natural transformations from GR to strict GR (in application of Proposition 6.7) by integration over moduli spaces.

7.1 Example from 2d TQFTs

We consider 2d TQFTs in the category $\mathrm{Vect}_{\mathbb{C}}$ of finite-dimensional complex vector spaces to illustrate the idea – which is standard and elementary – but one may try to adapt a similar construction for more fancy, higher-categorical notions of TQFTs.

A 2d TQFT is equivalent to the data of a unital Frobenius algebra $\mathcal{A}$, *i.e.* an object in $\mathrm{Vect}_{\mathbb{C}}$ equipped with a pairing and a commutative associative product which is invariant for the pairing – see *e.g.* [7, 1]. Using the pairing, we have canonical identifications between $\mathcal{A}$ and $\mathcal{A}^*$, and therefore distinguished elements

$$b^\dagger \in \mathrm{Hom}(\mathrm{Sym}^2(\mathcal{A}^*),\mathbb{C}), \qquad \mu \in \mathrm{Sym}^3(\mathcal{A}^*)\,,$$

representing the pairing and the product. The TQFT amplitudes are the element

$$F_{g,n} \in \mathrm{Hom}(\mathcal{A}^{\otimes n},\mathbb{C})$$

defined for any (g,n) as follows. Take a pair of pants decomposition $(P_1,\ldots,P_{2g-2+n})$ of a connected surface of genus g with n boundaries, and form the tensor product

$$\left(\bigotimes_{i=1}^{2g-2+n} \mu\right) \in (\mathcal{A}^*)^{6g-6+3n}.$$

Then, each time P_i has a common boundary component with P_j, we use the pairing to contract a copy of $\mathcal{A}$ in the i-th factor together with a copy of $\mathcal{A}$ in the j-th factor. This makes a total of

$3g-3+n$ pairings, so we are left with an element of $(\mathcal{A}^*)^{\otimes n} = \mathrm{Hom}(\mathcal{A}^{\otimes n}, \mathbb{K})$ which is by definition $F_{g,n}$. The axioms of a Frobenius algebra justify that the result is independent of the choice of pair of pants decomposition, and thus only depends on g and n, and is symmetric under permutation of the n remaining factors of $\mathcal{A}$.

STRICT SYMMETRIC TARGET THEORY. $\mathrm{Vect}_{\mathbb{C}}$ is a rather simple full subcategory of our category $\mathcal{C}$. For each stable (g,n), we choose $\underline{E}(g,n) = \mathrm{Hom}(\mathcal{A}^{\otimes n}, \mathbb{K})$. The union map comes from the tensor product (in this case the functor $\underline{E}$ is monoidal) and the morphisms are composition of tensors – using as many times as necessary the canonical identification $\mathcal{A} \cong \mathcal{A}^*$.

INITIAL DATA. We take $A = B = C = \mu$ representing the product. We also take $D = \langle H, \cdot \rangle$ where $H \coloneqq \sum_i e_i^2$ in terms of an orthogonal basis $(e_i)_i$. The four relation characterising a symmetric initial data are satisfied because, in each of them, each of the three terms in the left-hand side are separately symmetric under the permutation $\sigma_{1,2}$ due to the properties of the product.

THE AMPLITUDES. With this initial data, it is easy to see that $\varpi_{1,1} = F_{1,1} = D$, and to prove recursively that, if one writes the $\varpi_{g,n}$ as sum over the set $\mathbb{G}_{g,n}$ of fatgraphs described in Section 3.5, each term uniquely defines a topological class of pair of pants decomposition, and the value assigned to each fatgraph is by construction the TQFT amplitude $F_{g,n}$ – computed with help of this pair of pants decomposition. So, for any $2g-2+n > 0$, the strict GR amplitudes are

$$\varpi_{g,n} = |\mathbb{G}_1^{g,n}|\, F_{g,n}.$$

7.2 Example from quantum Airy structures

Another class of examples of strict symmetric GR is provided by the approach of Kontsevich-Soibelman approach to topological recursion [37, 2].

We first outline this theory. Let V be a $\mathbb{K}$-vector space, which we here assume to be finite-dimensional for simplicity, but one could easily handle infinite-dimensionality if equipped with filtrations by finite-dimensional subspaces. Let $\mathcal{W}_V^{\hbar}$ be the Weyl algebra of V, *i.e.* the unital algebra generated over $\mathbb{K}[[\hbar]]$ by $T^*V = V \oplus V^*$ with relations

$$\forall (v,\lambda) \in V \times V^*, \qquad [v,\lambda] = \hbar\, \lambda(v) \in \mathbb{K}[[\hbar]].$$

Definition 7.1 *A* quantum Airy structure *is the data of a linear map $L : V \to \mathcal{W}_V^{\hbar}$ such that there exists a basis $(x_i)_{i\in I}$ of linear coordinates on V in which L takes the form*

$$L_i = \hbar\partial_{x_i} - \sum_{a,b\in I} \tfrac{1}{2}A^i_{a,b}x_a x_b - \hbar B^i_{a,b}x_a\partial_{x_b} - \tfrac{\hbar^2}{2}C^i_{a,b}\partial_{x_a}\partial_{x_b} - \hbar D^i\,,$$

and satisfies Lie algebra relations

$$\forall (i,j) \in I^2, \qquad [L_i, L_j] = \sum_{a\in I} f^a_{i,j}\, L_a \tag{32}$$

for some scalars $A^i_{j,k} = A^i_{k,j}$, $B^i_{j,k}$, $C^i_{j,k} = C^i_{k,j}$, D^i and $f^k_{i,j} = -f^k_{j,i}$.

This notion is closely related to the quantisation of Lagrangians in T^*V which are osculating to the zero section and defined by quadratic equations. The constraints (32) imposes relations on the coefficients of L.

Lemma 7.2 *[2] Equation (32) is equivalent to the system of equations indexed by $i,j,k,\ell \in I$*

- $A^i_{j,k} = A^j_{i,k}$.
- $f^k_{i,j} = B^i_{j,k} - B^j_{i,k}$.
- $\sum_{a\in I} B^i_{j,a}A^a_{k,\ell} + B^i_{k,a}A^j_{a,\ell} + B^i_{\ell,a}A^j_{a,k} = (i \leftrightarrow j)$.
- $\sum_{a\in I} B^i_{j,a}B^a_{k,\ell} + B^i_{k,a}B^j_{a,\ell} + C^i_{\ell,a}A^j_{a,k} = (i \leftrightarrow j)$.
- $\sum_{a\in I} B^i_{j,a}C^a_{k,\ell} + C^i_{k,a}B^j_{a,\ell} + C^i_{\ell,a}B^j_{a,k} = (i \leftrightarrow j)$.
- $\sum_{a\in I} B^i_{j,a}D^a + \frac{1}{2}\sum_{a,b\in I} C^i_{a,b}A^j_{a,b} = (i \leftrightarrow j)$.

The third, fourth and fifth relation have the same index structure, and take the form of H-I-X relations. This was explained from a Lie algebraic perspective in [2]. The structure of the axioms for the initial data of a symmetric GR was modelled on these relations, and give them a geometric content as coming from different pair of pants decomposition of a sphere with four boundaries. In fact, we have chosen to define the BB-CA relation (23) to stick with Lemma 7.2, instead of the (29), which appeared in Remark 5.4 as an alternative sufficient condition to prove invariance under braidings of the symmetric GR.

Remark 7.3 *These relations implied by the quantum Airy structure are the motivation for choosing to define the BB-CA relation instead of the relation 29 for defining symmetric initial data. Indeed, only the BB-CA relation imply this set of equations by going to the strict case.*

We can also assemble the coefficients of a quantum Airy structure into tensors

$$A \in \mathrm{Hom}(V^{\otimes 3}, 1), \qquad B \in \mathrm{Hom}(V\otimes V, V), \qquad C \in \mathrm{Hom}(V, V\otimes V), \qquad D \in \mathrm{Hom}(V, \mathbb{K})$$

and the relations in Lemma 7.2 are then tensorial relations, where the sum over intermediate indices are replaced by composition of linear maps.

To any quantum Airy structure, one may associates amplitudes $F_{g,n} \in \mathrm{Hom}(V^{\otimes n}, \mathbb{K})$ in the following way.

Theorem 7.4 *Let L be a quantum Airy structure. There exists a unique $F \in \hbar^{-1}(\mathrm{Sym}V^*)[[\hbar]]$ without constant term, which we can decompose into*

$$F = \sum_{\substack{g\geq 0 \\ n\geq 1}} \frac{\hbar^{g-1}}{n!} F_{g,n}, \qquad F_{g,n} \in \mathrm{Sym}^n V^*\,,$$

and such that $F_{0,1} = 0$, $F_{0,2} = 0$ and

$$\forall i \in I, \qquad L_i \cdot \exp(F) = 0.$$

In fact, $F_{g,n}$ is uniquely determined by the initial data $F_{0,3} = A$ and $F_{1,1} = D$, and a recursion on $2g-2+n>0$. If we denote $(e_i)_{i\in I}$ the basis of V mentioned in Definition 7.1, and

$$F_{g,n}[i_1,\ldots,i_n] := F_{g,n}(e_{i_1}\otimes\cdots\otimes e_{i_n}),$$

this recursion takes the form

$$\begin{aligned} F_{g,n}[i_1,\ldots,i_n] \quad = \quad & \sum_{m=2}^{n}\sum_{a\in I} B^{i_1}_{i_m,a}F_{g,n-1}[a, i_{[\![2,n]\!]\setminus\{m\}}] \qquad (33) \\ & + \sum_{a,b\in I}\tfrac{1}{2}C^{i_1}_{a,b}\bigg(F_{g-1,n+1}[a,b,i_{[\![2,n]\!]}] + \sum_{\substack{J\cup J'=\{i_2,\ldots,i_n\} \\ h+h'=g}} F_{g,1+|J|}[a,J]F_{g',1+|J'|}[b,J']\bigg). \end{aligned}$$

exp(F) is sometimes called the partition function or wave function, and this theorem characterises it as the solution of differential constraints forming a Lie algebra. The only point which may not be obvious is the existence of F – which amounts to prove symmetry of the $F_{g,n}$ obtained by the recursion (33). This theorem is proved by general holonomicity argument in [37], and by direct computation from Lemma 7.2 in [2].

Target theory. For any stable (g,n), we put $\underline{E}(g,n) = \mathrm{Hom}(V^{\otimes n}, \mathbb{K})$. The union map is provided by the tensor product, and the glueing map by the composition of linear maps.

The amplitudes. The constraints on the tensors (A,B,C,D) described in Lemma 7.2 are exactly the ones characterising symmetric initial data. This is in fact the reason why we have postulated these relations in Definition 6.8, as well as their non-strict counterpart in Definition 5.1. The proof of Theorem 5.3 (of which Proposition 6.9 was a direct consequence) follows the same scheme as the proof of symmetry of $F_{g,n}$ in [2], except that it incorporates homotopy class considerations (which in the strict version are absent). Comparison of (33) with Definition 6.3 shows directly that $F_{g,n} = \varpi_{g,n}$.

7.3 Example from spectral curves

We now come back to the original topological recursion of [26] and show it provides an example of strict symmetric GR. Although it is known from [37, 2] that it is a particular case of the formalism of Kontsevich-Soibelman, its presentation of the original topological recursion is usually different as it involves the geometry of complex curves instead of algebras. We think it is also instructive to explain directly in the language of spectral curves how one can propose a strict target theory for which the topological recursion will coincide with the strict GR.

The starting point in [23, 26] is the data of

- a spectral curve, *i.e.* a compact Riemann surface Σ equipped with two meromorphic functions x, y, such that $x : S \to \mathbb{P}^1$ is a simple branched cover. We denote $\mathfrak{r}$ the divisor of the zeroes of $\mathrm{d}x$. We assume that $\mathrm{d}y$ has no zeroes on $\mathfrak{r}$. $\Delta := \{(z,z) \in S^2 \mid z \in S\}$ the diagonal divisor, and $\omega_{0,1} := y\mathrm{d}x$.
- a fundamental bidifferential of the second kind, *i.e.* $\omega_{0,2} \in H^0\big(S^2, K_S^{\boxtimes 2}(2\Delta)\big)^{\mathfrak{S}_2}$ such that the biresidue of $\omega_{0,2}$ at Δ is 1.

Let $U \subset S$ be a small enough neighbourhood of $\mathfrak{r}$, equipped with the holomorphic involution transformation $\varsigma : U \to U$ which exchange the sheets meeting at $r \in \mathfrak{r}$. We introduce the recursion kernel

$$K(z_1, z) := \frac{1}{2} \frac{\int_{\varsigma(z)}^{z} \omega_{0,2}(z_1, \cdot)}{\big(y(z) - y(\varsigma(z))\big)\mathrm{d}x(z)}.$$

The topological recursion of [26] is by definition

$$\begin{aligned} \omega_{g,n}^{\mathrm{TR}}(z_1, \ldots, z_n) \quad &= \quad \operatorname*{Res}_{z \to \mathfrak{r}} K(z_1, z)\bigg(\omega_{g-1,n+1}(z, \varsigma(z), z_2, \ldots, z_n) \\ &\quad + \sum_{\substack{h+h'=g \\ J \cup J' = [\![2,n]\!]}}^{\mathrm{stable}} \omega_{h,1+|J|}\big(z, (z_j)_{j\in J}\big) \otimes \omega_{h',1+|J'|}\big(\varsigma(z), (z_j)_{j \in J'}\big)\bigg), \end{aligned} \tag{34}$$

where $(z_j)_{j\in J}$ and $(z_j)_{j\in J'}$ are ordered tuples of variables in agreement with the ordering of J and J' as subsets of $[\![2,n]\!]$. It is known that $\omega_{g,n}$ inductively defined in this way for $2g-2+n > 0$ are such that

$$\omega_{g,n} \in H^0\big(S^n, K_S((6g-6+4n)\mathfrak{r})^{\boxtimes n}\big)^{\mathfrak{S}_n}. \tag{35}$$

Formula (34) already resembles strongly (30), so we just have to rephrase it in the language of the strict geometric recursion.

Target theory. For each stable (g,n), we put

$$\underline{E}(g,n) := \begin{cases} H^0\big(S^3,(K_S(2\mathfrak{r}+\mathfrak{p}))^{\boxtimes 3}\big) & \text{if } (g,n)=(0,3) \\ H^0\big(S^n,(K_S((6g-6+4n)\mathfrak{r})^{\boxtimes n})\big) & \text{otherwise} \end{cases} \tag{36}$$

where $\mathfrak{p}$ is the divisor of poles of $\mathrm{d}x$. The cutoff of the order of poles is not essential, but it makes $\underline{E}(g,n)$ finite-dimensional and thus simplifies our exposition. In particular, continuous multilinear maps $\prod_{i\in I} V_i \to W$ can be traded for linear maps $\bigotimes_{i\in I} V_i \to W$. We needed to take a larger space $\underline{E}(0,3)$ than appearing in (35), since it should host A, B and C. We will see that this choice allows to circumvent the fact that $K(z_0,z)$ is only defined for z in a neighbourhood of $\mathfrak{r}$, and the presence of a pole at coinciding points in $\omega_{0,2}(z,z')$. For $(g_i,n_i)_{i=1}^m$, we just take

$$\underline{E}\big((g_i,n_i)_{i\in I}\big) := \bigotimes_{i=1}^m \underline{E}(g_i,n_i)$$

with the identity as union map.

Glueing maps. Let $v=v(z_1,z_2,z_3)\in\underline{E}(0,3)$, and for each $\kappa\in\mathfrak{K}(g,n)$, w_κ be an element of $\underline{E}'(\kappa)$. We take

$$\begin{aligned}
\Theta_{(g-1,n+1)}(v,w_{g-1,n+1})(z_1,\ldots,z_n) &:= \operatorname*{Res}_{z\to\mathfrak{r}} \frac{2\,K(z_1,z)\,v(z_1,z,\varsigma(z))}{\mathrm{d}x(z_1)\big(\mathrm{d}x(z)\big)^3}\, w_{g-1,n+1}(z,\varsigma(z),z_2,\ldots,z_n)\,, \\
\Theta_{(g,n-1)_i}(v,w_{g,n-1}) &:= \operatorname*{Res}_{z\to\mathfrak{r}} \frac{K(z_1,z)\,v(z_1,z_i,z)}{\mathrm{d}x(z_1)\mathrm{d}x(z_i)\mathrm{d}x(z)}\big(\omega_{0,2}(z_i,z)w_{g,n-1}(\varsigma(z),z_2,\ldots,\widehat{z_i},\ldots,z_n) \\
&\qquad + \omega_{0,2}(z_i,\varsigma(z))w_{g,n-1}(z,z_2,\ldots,\widehat{z_i},\ldots,z_n)\big)\,, \\
\Theta_{(h,J),(h',J')}(v,w_{h,|J|},w_{h',|J'|}) &:= \operatorname*{Res}_{z\to\mathfrak{r}} \frac{K(z_1,z)\,v(z_1,z,\varsigma(z))}{\mathrm{d}x(z_1)\big(\mathrm{d}x(z)\big)^3}\big(w_{h,|J|}(z,(z_j)_{j\in J})w_{h',|J'|}(\varsigma(z),(z_j)_{j\in J'}) \\
&\qquad + w_{h,|J|}(\varsigma(z),(z_j)_{j\in J})w_{h,|J'|}(z,(z_j)_{j\in J'})\big)\,.
\end{aligned}$$

It is easy to check that these glueing maps indeed take values in $\underline{E}(g,n)$, and satisfy the axioms of a symmetric strict target theory.

Initial data. We take

$$\begin{aligned}
A(z_1,z_2,z_3) &:= \operatorname*{Res}_{z\to\mathfrak{r}} \frac{\omega_{0,2}(z_1,z)\omega_{0,2}(z_2,z)\omega_{0,2}(z_3,z)}{\mathrm{d}y(z)\mathrm{d}x(z)}, \\
B(z_1,z_2,z_3) &:= \mathrm{d}x(z_1)\mathrm{d}x(z_2)\mathrm{d}x(z_3), \\
C(z_1,z_2,z_3) &:= \mathrm{d}x(z_1)\mathrm{d}x(z_2)\mathrm{d}x(z_3), \\
D(z_1) &:= \operatorname*{Res}_{z\to\mathfrak{r}} \frac{1}{2}\,\frac{\int_{\varsigma(z)}^z \omega_{0,2}(\cdot,z_1)}{\big(y(z)-y(\varsigma(z))\big)\mathrm{d}x(z)}\,\omega_{0,2}(z,\varsigma(z)).
\end{aligned}$$

$A=\omega_{0,3}$ is an element of $H^0\big(S^3,(K_S^3(2\mathfrak{r}))^{\boxtimes 3}\big)^{\mathfrak{S}_3}\subset\underline{E}(0,3)^{\mathfrak{S}_3}$. The choice of B and C is *ad hoc* and all the non-trivial part about them has been transferred to the glueing maps they are always used with. $D=\omega_{1,1}$ is an element of $H^0\big(S,K_S(4\mathfrak{r})\big)$.

Lemma 7.5 *[2, Section 4.4] These (A,B,C,D) satisfy the 4 relations making them a symmetric initial data.*

Almost by construction, we have

Proposition 7.6 *For stable* (g,n), $\omega_{g,n}^{\mathrm{TR}}$ *coincide with the strict symmetric GR amplitudes* $\varpi_{g,n}$ *with target theory and initial data as above.* ∎

It is easy to see that the variants of the topological recursion already existing in the literature and motivated by applications – base curve $\mathbb{C}$ or $\mathbb{C}^*$ and S non-compact in Hurwitz theory [13, 9, 22], Gromov-Witten theory [12, 27, 28, 29], knot theory [15, 16, 8], statistical mechanics on the random lattice [10], base curve of higher genus in the context of Hitchin fibration [44], local spectral curves in [11], all fit in the framework of the strict symmetric GR as well.

7.4 From spectral curves to quantum Airy structures

The formalism of Section 7.3 is a particular case of the formalism of Section 7.2 by decomposing the multidifferentials $\omega_{g,n}(z_1,\ldots,z_n)$ in a basis of meromorphic 1-forms. We review this correspondence as it allows us a new observation, whose meaning will be questioned in Section 10.5.

Although we restrict to the case where S has only one ramification point to simplify the exposition, the discussion would also be valid for several ramification points.

By changing the local coordinate, we can always assume that the ramification point is located at $z=0$, and that the branched cover map is locally $x=\frac{z^2}{2}+x_0$ for some constant x_0. We introduce for $k\geq 0$

$$\xi^*(z)=\frac{z^{2k+1}}{2k+1}\,,$$

and define the family of odd meromorphic 1-forms $(\xi_k)_{k\geq 0}$ by

$$\xi_k(z_0)=\operatorname*{Res}_{z\to 0}\Big(\int_0^z\omega_{0,2}(\cdot,z_0)\Big)\frac{(2k+1)\mathrm{d}z}{z^{2k+2}}\,.$$

As a matter of fact, we have

$$\xi_k(z)=\frac{(2k+1)\mathrm{d}z}{z^{2k+2}}+O(\mathrm{d}z)\,,$$

and the expansion near the ramification point

$$\omega_{0,2}^{\mathrm{odd}}(z_1,z_2):=\tfrac{1}{2}\big(\omega_{0,2}(z_1,z_2)-\omega_{0,2}(-z_1,z_2)\big)=\sum_{k\geq 0}\mathrm{d}\xi_k^*(z_1)\,\xi_k(z_2)\,. \tag{37}$$

We remark that $\omega_{0,2}^{\mathrm{odd}}$ has double poles at $z_1=\pm z_2$, without residues, and one can show *via* direct computation of the Taylor expansion near 0 that it is still symmetric in its two variables.

We also introduce

$$\Upsilon(z)=\frac{-2\,z^2\mathrm{d}z}{\omega_{0,1}(z)-\omega_{0,1}(-z)}=\sum_{k\geq 0}\Upsilon_{2k}z^{2k}\,,$$

and give a name to the leading coefficient in ξ_0

$$\xi_0(z)=\frac{\mathrm{d}z}{z^2}+\xi_{0,0}\mathrm{d}z+O(z\mathrm{d}z)\,.$$

Let us define for $i,j,k\geq 0$ the scalars

$$\begin{aligned}
A^i_{j,k} &= \operatorname*{Res}_{z\to 0}\ \xi_i^*(z)\,\mathrm{d}\xi_j^*(z)\,\mathrm{d}\xi_k^*(z)\,\frac{\Upsilon(z)}{z^2\mathrm{d}z}\,,\\
B^i_{j,k} &= \operatorname*{Res}_{z\to 0}\ \xi_i^*(z)\,\mathrm{d}\xi_j^*(z)\,\xi_k(z)\,\frac{\Upsilon(z)}{z^2\mathrm{d}z}\,,\\
C^i_{j,k} &= \operatorname*{Res}_{z\to 0}\ \xi_i^*(z)\,\xi_j(z)\,\xi_k(z)\,\frac{\Upsilon(z)}{z^2\mathrm{d}z}\,,\\
D^i_{j,k} &= \delta_{k,0}\big(\xi_{0,0}\Upsilon_0+\tfrac{1}{8}\Upsilon_2\big)+\delta_{k,1}\,\tfrac{\Upsilon_0}{24}\,.
\end{aligned}$$

The following comparison statement is proved in [37, 2].

Theorem 7.7 *(A, B, C, D) defines a quantum Airy structure, and its amplitudes $F_{g,n}$ (Theorem 7.4) are the coefficients of decomposition of $\omega_{g,n}$ defined in (34)*

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{\substack{k_1,\ldots,k_n \geq 0 \\ \sum_i k_i \leq 3g-3+n}} F_{g,n}[k_1, \ldots, k_n] \prod_{i=1}^{n} \xi_{k_i}(z_i)\,.$$

We now come to our new observation, which we will return to in Section 10.5. Let us transform the tensors (A, B, C) into multidifferentials. Taking advantage of (37), we define

$$\begin{aligned}
\hat{A}(z_1, z_2, z_3) &= \sum_{i,j,k \geq 0} A^i_{j,k}\, \xi_i(z_1)\xi_j(z_2)\xi_k(z_3)\,, \\
\hat{B}(z_1, z_2, z_3) &= \sum_{i,j,k \geq 0} B^i_{j,k}\, \xi_i(z_1)\xi_j(z_2)\mathrm{d}\xi^*_k(z_3)\,, \\
\hat{C}(z_1, z_2, z_3) &= \sum_{i,j,k \geq 0} C^i_{j,k}\, \xi_i(z_1)\mathrm{d}\xi^*_j(z_2)\mathrm{d}\xi^*_k(z_3)\,.
\end{aligned}$$

Lemma 7.8 *We have the relation*

$$\hat{B}(z_1, z_2, z_3) + \hat{B}(z_1, z_3, z_2) = \hat{A}(z_1, z_2, z_3) + \hat{C}(z_1, z_2, z_3)\,.$$

Proof. We deduce from the definitions of (A, B, C) that

$$\begin{aligned}
\hat{A}(z_1, z_2, z_3) &= \operatorname*{Res}_{z \to 0} \Big(\int_0^z \omega^{\mathrm{odd}}_{0,2}(\cdot, z_1) \Big) \omega^{\mathrm{odd}}_{0,2}(z, z_2)\, \omega^{\mathrm{odd}}_{0,2}(z, z_3)\, \Upsilon(z)\,, \\
\hat{B}(z_1, z_2, z_3) &= \operatorname*{Res}_{z \to 0, \pm z_3} \Big(\int_0^z \omega^{\mathrm{odd}}_{0,2}(\cdot, z_1) \Big) \omega^{\mathrm{odd}}_{0,2}(z, z_2)\, \omega^{\mathrm{odd}}_{0,2}(z, z_3)\, \Upsilon(z)\,, \\
\hat{C}(z_1, z_2, z_3) &= \operatorname*{Res}_{z \to 0, \pm z_2, \pm z_3} \Big(\int_0^z \omega^{\mathrm{odd}}_{0,2}(\cdot, z_1) \Big) \omega^{\mathrm{odd}}_{0,2}(z, z_2)\, \omega^{\mathrm{odd}}_{0,2}(z, z_3)\, \Upsilon(z)\,,
\end{aligned}$$

where the poles at $\pm z_2$ and $\pm z_3$ occur owing to the ordering condition for the variables in $\omega^{\mathrm{odd}}_{0,2}$. We see that the contributions of the various residues satisfy the claimed relation. ■

In particular, as $\hat{A}(z_1, z_2, z_3)$ is a symmetric function of its three variables, Lemma 7.8 may account for the (so far empirical) observation that in computing with TR, the symmetrisation of some terms in the residue formula (34) seemed to reconstruct some other terms coming from other parts of the formula.

Part II

Geometric recursion with the topology of Teichmüller spaces

We shall now produce a number of interesting examples of targets. Typically, depending on the geometric framework we consider, we may add extra conditions for the nature of the objects in the image of **E**.

8 Teichmüller theory background

8.1 Teichmüller spaces

Let Σ be a bordered surface. If we consider the space of smooth metrics modulo conformal equivalence on Σ, we obtain a space of conformal classes of metrics on Σ, which is a $\mathrm{Diff}_0(\Sigma)$ fiber bundle over Teichmüller space. Here $\mathrm{Diff}_0(\Sigma)$ is the group of diffeomorphisms of Σ, which are isotopic to the identity (see e.g. [18]).

Recall that for Σ stable, $\mathcal{T}_\Sigma$ is in bijection with the set of hyperbolic metric on Σ, for which the boundaries are geodesic, modulo diffeomorphisms which are isotopic to identity. We further recall that $\mathcal{T}_\Sigma$ is a smooth manifold of dimension $6g-6+3n$, where g is the genus of Σ and n is the number of boundary components of Σ.

We denote $\mathcal{T}_{\partial\Sigma} = \mathbb{R}_{>0}^{\pi_0(\partial\Sigma)}$ the Teichmüller space associate to the boundary, and $p_\partial : \mathcal{T}_\Sigma \to \mathcal{T}_{\partial\Sigma}$ the induced restriction map. If L is an assignment of positive real numbers to the boundary components of Σ, we denote $\mathcal{T}_\Sigma(L) = p_\partial^{-1}(L)$.

We can of course also describe the Teichmüller space in terms of complex structures on Σ, e.g. $\mathcal{T}_\Sigma$ is also the set of equivalence classes of diffeomorphisms μ from Σ to a bordered Riemann surface $\mathcal{S}$. If $\mu_i : \Sigma \to \mathcal{S}_i$ with $i = 1, 2$ are two diffeomorphisms as above, we declare them equivalent if there exists a biholomorphic map $\Phi : \mathcal{S}_1 \to \mathcal{S}_2$, such that $\mu_2^{-1} \circ \Phi \circ \mu_1$ is isotopic to identity on Σ among such diffeomorphisms. When Σ is stable, the mapping class group Γ_Σ acts on the Teichmüller space $\mathcal{T}_\Sigma$ properly discontinuously, and the quotient is the moduli space $\mathcal{M}_\Sigma$.

To any bordered surface Σ there is naturally associated a closed marked surface $\tilde{\boldsymbol{\Sigma}}$, where each boundary component of Σ is collapsed to a point. Let P_∂ denote the resulting set of marked points on $\tilde{\boldsymbol{\Sigma}}$. The Teichmüller space of a marked closed surface, such as $\tilde{\boldsymbol{\Sigma}}$, $\mathcal{T}_{\tilde{\boldsymbol{\Sigma}}}$, is the space of complete finite volume hyperbolic metrics on $\tilde{\Sigma} - P_\partial$ modulo the diffeomorphisms of $\tilde{\Sigma}$, which induces the identity on the finite set of marked points P_∂ and which is isotopic to the identity modulo such. This Teichmüller space is also a manifold of dimension $6g-6+2n$ and its quotient by the mapping class group $\Gamma_{\tilde{\boldsymbol{\Sigma}}}$ is the usual moduli space $\mathcal{M}_{\tilde{\boldsymbol{\Sigma}}}$ of $|P_\partial|$-pointed Riemann surfaces of genus $g(\tilde{\Sigma})$.

8.2 Hyperbolic lengths and Teichmüller distance

If $\sigma \in \mathcal{T}_\Sigma$ is a hyperbolic metric and γ is a simple closed curve, there is a unique shortest geodesic in the homotopy class of γ, and we denote $\ell_\sigma(\gamma)$ its hyperbolic length. The length of a multicurve is by definition the sum of lengths of its components. We recall the famous collar lemma, see *e.g.* [14, Section 4.1]

Lemma 8.1 *Let Σ be a stable bordered surface, and σ a hyperbolic metric on Σ. Let $\gamma, \gamma' \subseteq \Sigma$ be two geodesics, which intersect transversally, and assume γ is simple and closed. If γ is contained in the interior of Σ and γ' is closed, then*

$$\sinh\Big(\frac{\ell_\sigma(\gamma)}{2}\Big)\sinh\Big(\frac{\ell_\sigma(\gamma')}{2}\Big) > 1.$$

If γ is a boundary of Σ, then

$$\sinh\Big(\frac{\ell_\sigma(\gamma)}{2}\Big)\sinh\big(\ell_\sigma(\gamma')\big) > 1.$$

The length spectrum is the sequence $(l_{\sigma,i})_{i\geq 1}$ of lengths of isotopy classes of simple closed curves (which are not isotopic to the boundary) in Σ°, in weakly increasing order. The systole is the shortest of these lengths $\mathrm{sys}_\sigma := l_{\sigma,1}$. We will exploit the following result.

Lemma 8.2 *If σ is a hyperbolic metric on a bordered surface Σ with non-zero boundary lengths, for any $t \in [0, \ell_\sigma(\partial\Sigma))$ there exists another hyperbolic metric $\tilde{\sigma}$ on Σ such that*

- $t = \ell_{\tilde{\sigma}}(\partial\Sigma) < \ell_\sigma(\partial\Sigma)$.
- $l_{\tilde{\sigma},i} \leq l_{\sigma,i}$ *for any* $i \geq 1$.
- $\mathrm{sys}_{\tilde{\sigma}} \geq \min\big(\mathrm{sys}_\sigma, 2\ln(1+\sqrt{2})\big)$.

Proof. The two first points can be found in [45, Theorem 3.3]. To get the last one, we need a small modification of its proof, which was communicated to us by Hugo Parlier. In this argument, by curve we mean geodesic representative of the homotopy class of a simple closed curve. We first remark that by the collar lemma, two curves of length $\leq 2\,\mathrm{arcsinh}(1) = 2\,\ln(1+\sqrt{2})$ cannot intersect. Let $\sigma(0)$ be a hyperbolic metric on Σ, and $\mathcal{S}_0$ be the set of systoles in $\Sigma_0 := \Sigma$. We denote Σ_0' the surface obtained by cutting Σ_0 along each of the curves which belong to $\mathcal{S}_0$. We denote

$$s_0 := \min\big(\mathrm{sys}_{\sigma(0)}, 2\,\ln(1+\sqrt{2})\big).$$

We are going to construct for each $t \in [0, \ell_\sigma(\partial\Sigma)]$ a finite set of curves $\mathcal{S}_t$ and hyperbolic metric $\sigma(t)$ on Σ such that

$$\ell_{\sigma(t)}(\partial\Sigma) = \ell_\sigma(\partial\Sigma) - t, \qquad \forall i \geq 1, \quad l_{\sigma(t),i} \leq l_{\sigma(0),i}. \tag{38}$$

In this process we will always denote Σ_t' the surface Σ_t cut along the curves in $\mathcal{S}_t$. We start to decrease the length as in [45, Theorem 3.3.] on each boundary component of the connected components of Σ_0' which does not belong to $\mathcal{S}_0$, so that the length spectrum of Σ_0 decreases continuously. This defines a new hyperbolic metric $\sigma(t)$ satisfying (38) for $t \geq 0$ small enough and we keep $\mathcal{S}_t := \mathcal{S}_0$. If for $t \in [0, \ell_\sigma(\partial\Sigma)]$ all curves which are not in $\mathcal{S}_t$ have $\sigma(t)$-length $> s_0$, the algorithm terminates. Otherwise, there exists a minimal $t^* \in (0, \ell_{\sigma(0)}(\partial\Sigma))$ such that some curve(s) in Σ has $\sigma(t^*)$-length s_0. We add all of those curves to $\mathcal{S}_t$ ($= \mathcal{S}_0$ if $t < t^*$) to form an updated set $\mathcal{S}_{t^*}$. We then continue to decrease the length of the boundary components of Σ_t' which are not elements of $\mathcal{S}_t$, continuously decreasing the $\sigma(t)$-length spectrum, and repeat our update each time we meet curves of $\sigma(t)$-length exactly s_0. For any t, the curves collected in $\mathcal{S}_t$ cannot intersect due to the previous observation, and there are at most $3g-3+n$ non-intersecting curves in the interior of a surface of genus g with n boundaries. Therefore, $|\mathcal{S}_t| \leq 3g-3+n$ for any t, and $\mathcal{S}$ can only be updated finitely many times. The construction makes sure that for all t, the last requirement $\mathrm{sys}_{\sigma(t)} \geq s_0$ holds, while (38) meets the two first requirements. ■

We also need a well-known estimate on the number of multicurves of bounded length, which we can make uniform in the boundary length using the previous theorem. The more precise asymptotic estimate of this number established by Mirzakhani will not be needed here.

Theorem 8.3 *Let Σ be a bordered surface of genus g with n boundaries. For any $\varepsilon > 0$, there exists a constant $M(g,n,\varepsilon)$ such that, for any $\sigma \in \mathcal{T}_\Sigma$, the number $\mathrm{N}_\sigma(L)$ of geodesic multicurves of length at most L on Σ satisfies the bound*

$$\forall L \geq 0, \qquad \mathrm{N}_\sigma(L) \leq M(g,n,\varepsilon)\Bigg(\prod_{\substack{\gamma \in \hat{S}_\Sigma^\circ \\ \ell_\sigma(\gamma) \leq \varepsilon}} \frac{1}{\ell_\sigma(\gamma)}\Bigg) L^{6g-6+2n}$$

In particular, if $\mathrm{sys}_\sigma \geq \varepsilon$*, the right-hand side is equal to* $M(g,n,\varepsilon)\, L^{6g-6+2n}$.

Proof. We rely on an intermediate result of Mirzakhani [42, Proposition 3.6]: if Σ_σ only has punctures (boundaries of length 0), for any $\varepsilon > 0$, there exists a constant $M(g,n,\varepsilon) > 0$ such that for any hyperbolic metric we have

$$\mathrm{N}_\sigma(L) \leq M(g,n,\varepsilon)\Bigg(\prod_{\substack{\gamma\in\hat{\mathcal{S}}^\circ_\Sigma \\ \ell_\sigma(\gamma)\leq\varepsilon}} \frac{1}{\ell_\sigma(\gamma)} \Bigg) L^{6g-6+2n}. \tag{39}$$

If σ is a hyperbolic metric on Σ with non-zero boundary lengths, Theorem 8.2 provides us with $\tilde{\sigma}$ having zero boundary lengths, such that $l_{i,\tilde{\sigma}} < l_{i,\sigma}$ for any $i \geq 1$. *A fortiori* we must have $\mathrm{N}_\sigma(L) \leq \mathrm{N}_{\tilde{\sigma}}(L)$ and $\mathrm{sys}_{\tilde{\sigma}} \geq 2\ln(1+\sqrt{2})$. The bound (39) now extends uniformly to hyperbolic metrics assigning positive lengths to the boundaries. ∎

If Σ has no boundaries and $\mathcal{S}_1$ and $\mathcal{S}_2$ are two Riemann surfaces diffeomorphic to Σ, a quasiconformal map is a smooth map $f : \mathcal{S}_1 \to \mathcal{S}_2$ such that $k(f) \coloneqq \sup_{\mathcal{S}_1} \left|\frac{\bar{\partial} f}{\partial f}\right| < 1$. Its dilation factor is $K(f) \coloneqq \frac{1+k(f)}{1-k(f)}$. If σ, σ' are two hyperbolic metrics on Σ, we let

$$d_T(\sigma,\sigma') \coloneqq \inf\left\{ \tfrac{1}{2}\ln K(f) \quad \middle| \quad f : \Sigma_\sigma \to \Sigma_{\sigma'} \text{ quasi-conformal}\right\}$$

and it defines a distance, called the Teichmüller distance, making $\mathcal{T}_\Sigma$ a complete metric space.

Remark 8.4 *If* $\varphi : \Sigma \to \Sigma'$ *is a morphism between surfaces in the category of bordered surfaces, then* φ *induces a continuous map* $\mathcal{T}_\Sigma \to \mathcal{T}_{\Sigma'}$ *which is an isometry for the respective Teichmüller distances.*

We use this distance to compare lengths of curves with respect to different metrics on Σ.

Theorem 8.5 *[55, Lemma 3.1]. Let* Σ *be a stable surface without boundaries, and* $\gamma \subset \Sigma$ *be a simple closed curve whose homotopy class is non trivial. Let* $\ell_\sigma(\gamma)$ *be the length of the geodesic (for a hyperbolic metric* σ*) representative in its homotopy class. Then for any two hyperbolic metrics* σ, σ' *on* Σ

$$e^{-2d_T(\sigma,\sigma')} \leq \frac{\ell_\sigma(\gamma)}{\ell_{\sigma'}(\gamma)} \leq e^{2d_T(\sigma,\sigma')}$$

The result is also true if Σ *has boundaries, provided we use the Teichmüller distance on* $\mathcal{T}_{\Sigma^d}$*, where recall that* Σ^d *is the surface without boundary obtained by doubling* Σ *along* $\partial\Sigma$*.*

8.3 Teichmüller space of pointed bordered surfaces

We will also need the Teichmüller space of pointed bordered surfaces. Let $\mathbf{\Sigma}$ be a pointed bordered surface, so we have marked points on the boundary $o = (o_b), b \in \pi_0(\partial\Sigma)$.

Definition 8.6 *The Teichmüller space* $\mathcal{T}_{\mathbf{\Sigma}}$ *for a pointed bordered surface* $\mathbf{\Sigma}$ *is the set of equivalence classes of diffeomorphisms* μ *from* Σ *to a bordered Riemann surface* $\mathcal{S}$*. The equivalence relation is as above, with the change, in the notation from above, that* $(\mu_2)^{-1} \circ \Phi \circ \mu_1$ *restricts to the identity on* o *and that isotopies are required to preserve this property.*

There is a canonical projection

$$p_{\mathbf{\Sigma}} : \mathcal{T}_{\mathbf{\Sigma}} \longrightarrow \mathcal{T}_\Sigma,$$

and it is an $\mathbb{R}^{\pi_0(\partial\Sigma)}$-bundle.

Recall that $\Delta_{\mathbf{\Sigma}}$ is the group generated by Dehn twists along the boundaries. Its action is free on $\mathcal{T}_{\mathbf{\Sigma}}$, and the induced projection

$$\tilde{p}_{\mathbf{\Sigma}} : \mathcal{T}_{\mathbf{\Sigma}}/\Delta_{\Sigma} \longrightarrow \mathcal{T}_{\Sigma}$$

is a $U(1)^{\pi_0(\partial\Sigma)}$-bundle. We denote $\mathcal{T}_{\mathbf{\Sigma}}/\Delta_{\Sigma}$ by $\widetilde{\mathcal{T}}_{\mathbf{\Sigma}}$.

When $\mathbf{\Sigma}$ is connected, has genus g with n boundaries (stability requires $2g-2+n>0$), $\widetilde{\mathcal{T}}_{\Sigma}$ is a smooth manifold of real dimension $6g-6+4n$. It will be implicit that all maps considered between Teichmüller spaces are smooth.

If P is a pair of pants, we get a canonical identification

$$\mathcal{T}_P \cong \mathbb{R}^3_{>0}$$

once we have chosen an ordering of the boundary components of P. If we choose an embedding of a uni-trivalent Y-graph into $(P,\partial P)$ (see figure ...) and we let o be the univalent vertices of Y and $\mathbf{P}=(P,o)$, then we get canonical identifications

$$\mathcal{T}_{\mathbf{P}} \cong (\mathbb{R}_{>0}\times\mathbb{R})^3, \qquad \widetilde{\mathcal{T}}_{\mathbf{P}} \cong (\mathbb{R}_{>0}\times U(1))^3 .$$

We denote by $(L_i,\alpha_i)_{i=1}^3$ the resulting coordinates on $\widetilde{\mathcal{T}}_{\mathbf{P}}$, and stress that they depend on the embedding of the above mentioned Y-graph.

8.4 The fibrations relevant for glueing and cutting

Let $\mathbf{\Sigma}$ be a stable pointed bordered surface, and γ be a stable oriented pointed multicurve in Σ. We introduce $\widetilde{\mathcal{T}}^{=}_{\mathbf{\Sigma}^\gamma} \subset \widetilde{\mathcal{T}}_{\mathbf{\Sigma}^\gamma}$, the submanifold along which the connected components of $\partial^+_\gamma\Sigma^\gamma$ and $\partial^-_\gamma\Sigma^\gamma$ have the same length and the image of p_∂ followed by the projection onto the factors $\pi_0(\partial^+_\gamma\Sigma^\gamma)$ and $\pi_0(\partial^-_\gamma\Sigma^\gamma)$ is contained in the diagonal. We let

$$\iota_\gamma : \mathcal{T}^{=}_{\mathbf{\Sigma}^\gamma} \longrightarrow \mathcal{T}_{\mathbf{\Sigma}^\gamma}$$

denote the inclusion map. The submanifold $\mathcal{T}^{=}_{\mathbf{\Sigma}^\gamma}$ is defined precisely so that we get a smooth fibration

$$\vartheta_\gamma : \mathcal{T}^{=}_{\mathbf{\Sigma}^\gamma} \longrightarrow \mathcal{T}_{\mathbf{\Sigma}},$$

obtained by using the glueing map which is induced from the cutting construction of $\mathbf{\Sigma}^\gamma$.

We can see that $\mathcal{T}^{=}_{\mathbf{\Sigma}^\gamma}$ is a principal $\mathbb{R}^{\pi_0(\gamma)}$-bundle over $\mathcal{T}_{\mathbf{\Sigma}}$. Suppose $\mu : \Sigma^\gamma \to \mathcal{S}$ represents a point in $\mathcal{T}^{=}_{\mathbf{\Sigma}^\gamma}$. Then $\alpha \in \mathbb{R}^{\pi_0(\gamma)}$ acts by precomposing μ with a diffeomorphism of Σ^γ, which is the identity on the complement of a small tubular neighbourhood of $\partial^+_\gamma\Sigma^\gamma \cup \partial^-_\gamma\Sigma^\gamma$, and which rotates by the amount $\ell_\sigma(\beta)\alpha(\beta)$ on each of the pairs of boundary components in Σ^γ corresponding to $\beta \in \pi_0(\gamma)$, measured with respect to the hyperbolic metric σ specified by μ. Note that the subgroup generated by simultaneous Dehn twists along $\partial^+_\gamma\Sigma^\gamma$ and $\partial^-_\gamma\Sigma^\gamma$ is identified with the subgroup $\mathbb{Z}^{\pi_0(\gamma)} \subset \mathbb{R}^{\pi_0(\gamma)}$.

The fibration ϑ_γ comes with the action of the exact sequence of groups (3)

$$1 \longrightarrow \Delta_\gamma \longrightarrow \Gamma_{\mathbf{\Sigma}^\gamma} \longrightarrow \mathrm{Stab}(\gamma) \longrightarrow 1, \qquad \Delta_\gamma := \prod_{\beta\in\pi_0(\gamma)} \langle \delta_{\beta^+}\delta_{\beta^-}^{-1}\rangle.$$

If we rather consider a subgroup $G_{\mathbf{\Sigma}^\gamma}$ of $\Gamma_{\mathbf{\Sigma}^\gamma}$, which contains $\prod_\beta\langle\delta_{\beta^+}\delta_{\beta^-}^{-1}\rangle$, then this is also true if we replace $\mathrm{Stab}(\gamma)$ by the image of $G_{\mathbf{\Sigma}^\gamma}$ in $\Gamma_{\mathbf{\Sigma}}$ via the projection map in (3). In any case, we denote

$$\tilde{q}_\gamma : \mathcal{T}^{=}_{\mathbf{\Sigma}^\gamma} \longrightarrow \mathcal{T}^{=}_{\mathbf{\Sigma}^\gamma}/\Delta_\gamma$$

the quotient map and

$$\tilde{\vartheta}_\gamma : \mathcal{T}_{\mathbf{\Sigma}^{=}_{\gamma}}/\Delta_\gamma \longrightarrow \mathcal{T}_{\mathbf{\Sigma}}$$

the unique glueing map such that $\tilde{\vartheta}_\gamma \circ \tilde{q}_\gamma = \vartheta_\gamma$.

It will be useful to consider the decorated Teichmüller space modding out by the action of all Dehn twists along the boundaries. In this context, we have an inclusion map

$$\tilde{\iota}_\gamma : \widetilde{\mathcal{T}}_{\mathbf{\Sigma}^{=}_{\gamma}} \longrightarrow \widetilde{\mathcal{T}}_{\mathbf{\Sigma}^{\gamma}}$$

and a glueing fibration

$$\tilde{\vartheta}_\gamma : \widetilde{\mathcal{T}}_{\mathbf{\Sigma}^{=}_{\gamma}} \longrightarrow \widetilde{\mathcal{T}}_{\mathbf{\Sigma}}. \tag{40}$$

Although these symbols are the same as for the maps when we do not divide by Dehn twists along the boundaries, which one of these maps we refer to with the symbols $\tilde{\iota}_\gamma$ and $\tilde{\vartheta}_\gamma$ will be clear from the context.

But to actually compute with the geometric recursion, we only use this fibration in the case where γ is the internal boundary of an embedded pair of pants $f : P \to \Sigma$ as in Section 2.5. In case **I** and **II**, γ has two components so $U(1)^2$ acts simply transitively on the fibers of $\tilde{\vartheta}_\gamma$, while in case **I'**, we rather have $U(1)$ acting simply transitively.

9 Continuous functions on Teichmüller spaces

9.1 On pointed bordered Teichmüller spaces

We shall describe several variants of target theories based on the space of continuous functions on Teichmüller spaces. We first present the case of continuous functions on $\widetilde{\mathcal{T}}_{\mathbf{\Sigma}}$, for which we can take advantage of the proper fibration $\tilde{\vartheta}_\gamma : \widetilde{\mathcal{T}}_{\mathbf{\Sigma}^{=}_{\gamma}} \to \widetilde{\mathcal{T}}_{\mathbf{\Sigma}}$ induced by the glueing map $\tilde{\vartheta}_\gamma$. The last paragraph in this section describes the small amendments of this construction which turns $\mathcal{C}^0(\mathcal{T}_{\mathbf{\Sigma}})$ into a target theory.

Spaces for the target theory

We let

$$E(\mathbf{\Sigma}) \coloneqq \mathcal{C}^0(\widetilde{\mathcal{T}}_{\mathbf{\Sigma}}) \cong \mathcal{C}^0(\mathcal{T}_{\mathbf{\Sigma}})^{\Delta_{\mathbf{\Sigma}}} \tag{41}$$

equipped with the topology of convergence on all compact subsets. This is a locally convex, Hausdorff, complete, topological vector space which is functorial in $\mathbf{\Sigma}$. We present it as a projective limit of such spaces as follows.

We introduce the directed set $\mathscr{I}_{\mathbf{\Sigma}} = \mathbb{R}_{>0}$. For any $\varepsilon \in \mathscr{I}_{\mathbf{\Sigma}}$, we introduce

$$K^{\circ}_{\mathbf{\Sigma}}(\varepsilon) \coloneqq \left\{\sigma \in \widetilde{\mathcal{T}}_{\mathbf{\Sigma}} \quad | \quad \mathrm{sys}_\sigma \geq \varepsilon\right\}, \qquad K^{\partial}_{\mathbf{\Sigma}}(\varepsilon) = \left\{\sigma \in \widetilde{\mathcal{T}}_{\mathbf{\Sigma}} \quad | \quad \forall b \in \pi_0(\partial\Sigma),\ \ell_\sigma(b) \geq \varepsilon\right\}.$$

Beware that the notation $\circ$ does not stand here for the interior, but rather reminds the reader that this set controls lengths of curves in the interior of Σ. We also introduce the ε-thick subset of $\mathcal{T}_{\mathbf{\Sigma}}$

$$K_{\mathbf{\Sigma}}(\varepsilon) \coloneqq K^{\circ}_{\mathbf{\Sigma}}(\varepsilon) \cap K^{\partial}_{\Sigma}(\varepsilon). \tag{42}$$

The sets $K^{\circ}_{\mathbf{\Sigma}}(\varepsilon)$ and $K^{\partial}_{\mathbf{\Sigma}}(\varepsilon)$ are stable under the action of the mapping class group, and allow arbitrarily large and small boundary lengths, and for this reason they are not compact. Note that if $\mathbf{P}$ is a pair of pants, $K_{\mathbf{P}}(\varepsilon) = K^{\partial}_{\mathbf{P}}(\varepsilon)$.

We then take

$$E^{\varepsilon}(\mathbf{\Sigma}) := \mathcal{C}^0(K_{\mathbf{\Sigma}}(\varepsilon)).$$

On this space, we define the functorial exhaustive family of seminorms indexed by the set $\mathscr{A}^{\varepsilon}_{\mathbf{\Sigma}}$ of compact subsets of $K_{\mathbf{\Sigma}}(\varepsilon)$, such that

$$\forall K \in \mathscr{A}^{\varepsilon}_{\mathbf{\Sigma}}, \qquad |f|_K := \sup_{\sigma \in K} |f(\sigma)|.$$

This gives $E^{\varepsilon}(\mathbf{\Sigma})$ the structure of a locally convex, Hausdorff, complete topological vector spaces, and we have continuous restriction maps $E^{\varepsilon}(\mathbf{\Sigma}) \to E^{\varepsilon'}(\mathbf{\Sigma})$ whenever $\varepsilon < \varepsilon'$. One then easily checks that $E(\mathbf{\Sigma})$ in (41) is the projective limit of these spaces over the directed set $\mathbb{R}_{>0}$. Recalling the notations of Definition 3.3, we also have seminorms

$$\|f\|_{\varepsilon} = \sup_{\sigma \in K_{\mathbf{\Sigma}}(\varepsilon)} |f(\sigma)|$$

and a subspace

$$E'(\mathbf{\Sigma}) = \left\{ f \in \mathcal{C}^0(\widetilde{\mathfrak{T}}_{\mathbf{\Sigma}}) \quad \middle| \quad \forall \varepsilon > 0, \quad \|f\|_{\varepsilon} < +\infty \right\}.$$

Length functions

For any $\varepsilon > 0$ and K a compact subset of $K_{\mathbf{\Sigma}}(\varepsilon)$, we use the hyperbolic length ℓ_σ to define the length functions,

$$\forall \gamma \in S_{\mathbf{\Sigma}}, \qquad l_K(\gamma) = \min_{\sigma \in K} \ell_\sigma(\gamma). \tag{43}$$

Since K is compact, Lemma 8.2 shows that for any $\sigma \in K$, there exists a constant $c_K \in (0,1)$ such that

$$c_K\, \ell_\sigma(\gamma) \leq l_K(\gamma).$$

As the systole is bounded below by construction on each $K_{\mathbf{\Sigma}}(\varepsilon)$, we deduce that the length functions (43) satisfy the lower bound axiom. We also deduce that the number of $\gamma \in S^{\circ}_{\Sigma}$ with $l_K \leq L$ is bounded from above by $\mathrm{N}_\sigma(L/c_K)$. Consequently, Theorem 8.3 guarantees they also satisfy the polynomial growth axiom. Note that, as $K_{\mathbf{\Sigma}}(\varepsilon)$ allows for arbitrarily large boundary lengths, it was important to obtain an estimate of the number of simple closed curves below a given hyperbolic length which is uniform with respect to the boundary lengths, and this was achieved thanks to Lemma 8.2.

Union morphisms

As union morphism, we take $\sqcup : \mathbf{E}(\Sigma_1) \times \mathbf{E}(\Sigma_2) \to \mathbf{E}(\Sigma_1 \cup \Sigma_2)$ given by $f_1 \sqcup f_2 = q_1^* f_1 \cdot q_2^* f_2$, where $q_i : \mathbf{E}(\Sigma_1 \cup \Sigma_2) \to \mathbf{E}(\Sigma_i)$ are the projections. It is clearly continuous.

Glueing morphisms

We now construct the glueing morphisms. With notations from the glueing axiom, if $(f_1, f_2) \in E(\mathbf{\Sigma}_1) \times E(\mathbf{\Sigma}_2)$ and $\iota_\gamma : \widetilde{\mathfrak{T}}^{=}_{\mathbf{\Sigma}^\gamma} \to \widetilde{\mathfrak{T}}_{\mathbf{\Sigma}^\gamma}$ denotes the inclusion, $\iota_\gamma^*(f_1 \sqcup f_2)$ is a continuous function on $\widetilde{\mathfrak{T}}^{=}_{\mathbf{\Sigma}^\gamma}$. We can then define

$$\mathbf{\Theta}_\gamma(f_1, f_2)(\sigma) := \int_{\tilde{\vartheta}_\gamma^{-1}(\sigma)} \iota_\gamma^*(f_1 \sqcup f_2)\, \mathrm{d}\alpha. \tag{44}$$

where $\mathrm{d}\alpha$ is the measure on the fibers of $\tilde{\vartheta}$ with total mass 1 and which is invariant under action of $\mathbb{R}^{\pi_0(\gamma)}$. Equation (44) gives a continuous function of $\sigma \in \widetilde{\mathfrak{T}}_{\mathbf{\Sigma}}$. Since $\tilde{\vartheta}_\gamma$ is a proper fibration, $\mathbf{\Theta}_\gamma$ is continuous. It is easily seen that the union and glueing maps satisfy all the compatibility requirements required for $\mathbf{E}$ to be a target theory.

Admissibility for initial data

Taking into account the uniform comparison from Lemma 8.5 for hyperbolic lengths for two metrics in any compact subset of Teichmüller space, the decay axiom on initial data (A,B,C,D) takes the following equivalent form.

We should assume that for any $s,\varepsilon>0$, there exists a constant $M(s,\varepsilon)>0$ such that, if $\mathbf{P}$ is a pair of pants with boundaries $\partial_- P = b_1$ and $\partial_+ P = b\cup b'$,

$$\sup_{\sigma\in K_{\mathbf{P}}(\varepsilon)} \ell_\sigma(b')^s \left|B^b_{\mathbf{P}}(\sigma)\right| \leq M(s,\varepsilon), \qquad \sup_{\sigma\in K_{\mathbf{P}}(\varepsilon)} (\ell_\sigma(b)+\ell_\sigma(b'))^s \left|C_{\mathbf{P}}(\sigma)\right| \leq M(s,\varepsilon)\,. \tag{45}$$

GR is well-defined

If (A,B,C,D) are admissible initial data for the target theory $\mathcal{C}^0(\widetilde{\mathcal{T}}_{\mathbf{\Sigma}})$ we have just described, Theorem 3.7 guarantees that GR constructs a well-defined functorial assignment $\mathbf{\Sigma}\mapsto\Omega_{\mathbf{\Sigma}}\in E'(\mathbf{\Sigma})$. Note in particular that $\Omega_{\mathbf{\Sigma}}$ is uniformly bounded when the boundary lengths go to $+\infty$ (provided the systole does not go to zero).

If we rather use initial data which are admissible with $\daleth$-decay at infinity (see Definition 4.1), Proposition 4.2 shows that $\Omega_{\mathbf{\Sigma}}$ can grow at most like $\prod_{\beta\in\pi_0(\partial\Sigma)}\daleth(\ell_\sigma(\beta))$ when the boundary lengths go to $+\infty$ (provided the systole remains bounded from below). There are no constraint on the choice of the function $\daleth:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$.

For pedagogical reasons, let us repeat, in outline, the proof of Theorem 3.7 for this particular target theory.

The idea is to start by proving – thanks to the decay axiom – the absolute convergence of the sums uniformly over any compact in Teichmüller space. Thanks to Lemma 8.5, it is obtained for instance by proving first pointwise convergence at a point $\sigma\in\widetilde{\mathcal{T}}_{\mathbf{\Sigma}}$ with help of an upper bound on the zeta function of σ-hyperbolic lengths coming from Theorem 8.3. Note that this upper bound in fact only depends on the systole of σ, and may diverge only when the systole approaches zero. As a result, the pointwise limit (and actually uniform limit over any compact according to the previous remark) is bounded by a constant that only depends on the systole, *i.e.* on our lower bound ε on the systole. This shows that the limit exists in $E'(\mathbf{\Sigma})\subset E(\mathbf{\Sigma})$. However we do not prove that the series converge uniformly over the systole subsets $K_{\mathbf{\Sigma}}(\varepsilon)$ – this would be wrong[4]. Indeed, all the terms in our infinite sums are related by the action of the mapping group. As the sets $K_{\mathbf{\Sigma}}(\varepsilon)$ are stable under this action, all the terms in our sum have equal $\|\cdot\|_\varepsilon$-norm, and therefore the sum of their $\|\cdot\|_\varepsilon$-norm is $+\infty$. Now that we have convergence on any compact subset, we are free to reshuffle the terms of the sum in an arbitrary way. As our sums range over mapping class group orbits, the limit must be mapping class group invariant, and more generally functorial.

9.2 On bordered Teichmuller spaces

A small adaptation of the previous case provides a target theory

$$\mathbf{E}(\Sigma)=\mathcal{C}^0(\mathcal{T}_\Sigma).$$

The only notable difference lies in the definition of glueing morphism. We first use the natural projections $\tilde{p}_{\mathbf{\Sigma}_i}:\widetilde{\mathcal{T}}_{\mathbf{\Sigma}_i}\to\mathcal{T}_{\Sigma_i}$ to lift any element $f\in\mathcal{C}^0\big(\prod_{i=1}^2\mathcal{T}_{\Sigma_i}\big)$ to an element $\tilde{f}\in\mathcal{C}^0\big(\prod_{i=1}^2\widetilde{\mathcal{T}}_{\mathbf{\Sigma}_i}\big)$, and apply the glueing morphism of the previous target theory. Almost by construction, the result is

[4] A prototype of this phenomenon is that the series $x\mapsto\sum_{n\in\mathbb{Z}}\exp(-(n+x)^2)$ converges uniformly on any compact to a 1-periodic continuous function of $x\in\mathbb{R}$, but this convergence is not uniform on $\mathbb{R}$.

a function which is constant over the fibers of $\tilde{p}_{\mathbf{\Sigma}}$, therefore drops to an element of $\mathcal{C}^0(\mathcal{T}_\Sigma)$ which we define to be $\mathbf{\Theta}(f)$. In other words

$$\mathbf{\Theta}(f_1, f_2)(\sigma) = f_1(\sigma_1) f_2(\sigma_2)$$

where $\sigma_i = \sigma|_{\Sigma_i}$. In this context, the GR formula (7) reads

$$\begin{aligned}\Omega_{\mathbf{\Sigma}}(\sigma) \quad = \quad & \sum_{b \in \pi_0(\partial_+\Sigma)} \sum_{c \in \mathfrak{C}_{b_1,b}(\mathbf{\Sigma})} B^b_{\mathbf{P}_c}(L_{b_1}, L_b, \ell_\sigma(\gamma_c))\, \Omega_{\mathbf{\Sigma}_c}(\sigma|_{\Sigma_c}) + \\ & + \frac{1}{2} \sum_{c \in \mathfrak{C}_{b_1,b_1}(\mathbf{\Sigma})} C_{\mathbf{P}_c}(L_1, \ell_\sigma(\gamma_c^1), \ell_\sigma(\gamma_c^2))\, \Omega_{\mathbf{\Sigma}_c}(\sigma|_{\Sigma_c}). \end{aligned} \tag{46}$$

Actually, the lift by the projection map $\tilde{p}_{\mathbf{\Sigma}}$ gives a natural transformation relating these target theories

$$\mathcal{C}^0(\mathcal{T}_\Sigma) \Longrightarrow \mathcal{C}^0(\widetilde{\mathcal{T}}_{\mathbf{\Sigma}})$$

and it is compatible with union and glueing morphisms. Following Section 3.4, we can use them to transport initial data and GR from one target theory to the other.

9.3 Coupling to 2d TQFTs

Let $(\mathcal{A}, \cdot, \langle\,,\,\rangle)$ be a Frobenius algebra, together with an (arbitrary) choice of hermitian norm $|\cdot|_{\mathcal{A}}$. The previous construction can be adapted to make

$$\mathbf{E}(\mathbf{\Sigma}) = \mathcal{C}^0\big(\mathcal{T}_\Sigma, \mathcal{A}^* \otimes \mathcal{A}^{\otimes \pi_0(\partial_+\Sigma)}\big) \qquad \text{or} \qquad \mathbf{E}(\mathbf{\Sigma}) = \mathcal{C}^0\big(\widetilde{\mathcal{T}}_{\mathbf{\Sigma}}, \mathcal{A}^* \otimes \mathcal{A}^{\otimes \pi_0(\partial_+\Sigma)}\big) \tag{47}$$

where the glueing morphism now incorporates the product in $\mathcal{A}$, and the norms of these spaces are provided by the supremum of $|\cdot|_{\mathcal{A}}$ on compact subsets.

Another way to describe (47) is to make the tensor product of the strict target theory based on the 2d TQFT of $\mathcal{A}$, with the aforementioned target theories of continuous functions over Teichmüller spaces. This tensor product is well-defined because $\mathcal{A}$ is a finite-dimensional vector space. When $\mathcal{A}$ is semi-simple, the choice of a canonical basis on $\mathcal{A}$ gives a natural choice of hermitian norm on $\mathcal{A}$. This is a basic example of the fibering procedure described later in Section 10.3.

9.4 Continuous functions of boundary lengths

We can define a strict, partial target theory based on a space of continuous functions of the boundary lengths. We fix a function $\daleth : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ which will control the allowed behaviour of our amplitudes when the boundary lengths approach 0 or $+\infty$.

Definition 9.1 *We introduce the space $\mathcal{C}^0_{\daleth\cdot\mathrm{poly}}(\mathbb{R}^n_{>0})$ be the space of continuous functions f on $\mathbb{R}^n_{>0}$ for which there exists $r > 0$ such that*

$$\sup_{L \in \mathbb{R}^n_{>0}} \frac{|f(L)|}{\prod_{i=1}^n (1 + L_i)^r \daleth(L_i)} < +\infty.$$

We define a (partial) strict target theory

$$\underline{E}(g, n) := \begin{cases} \mathcal{C}^0(\mathbb{R}^3_{>0}) & \text{if } (g,n) = (0,3) \\ \mathcal{C}^0_{\daleth\cdot\mathrm{poly}}(\mathbb{R}^n_{>0}) & \text{otherwise} \end{cases}$$

which we will use in the next paragraph.

The union morphisms are induced by the product map

$$\sqcup : \mathcal{C}^0_{\neg\cdot\mathrm{poly}}(\mathbb{R}^{n_1}_{>0}) \times \mathcal{C}^0_{\neg\cdot\mathrm{poly}}(\mathbb{R}^{n_2}_{>0}) \to \mathcal{C}^0_{\neg\cdot\mathrm{poly}}(\mathbb{R}^{n_1+n_2}_{>0}), \qquad f_1 \sqcup f_2 = f_1 \cdot f_2.$$

Here we see it is convenient to work with bilinear maps as we do in the category $\mathcal{C}$, rather than to turn it into a linear map via the tensor product, to avoid discussion of completion issues.

To complete the definition, we should introduce the glueing morphisms. If (g_1, n_1) and (g_2, n_2) are two stable objects which we glue along the last component of (g_1, n_1) and the first component of (g_2, n_2), we define their glueing by

$$\Theta(f_1, f_2) = \int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, f(L_1, \ldots, L_{n_1-1}, \ell)\, g(\ell, L_2', \ldots, L_{n_2}'). \tag{48}$$

The reason to include an extra factor of ℓ will appear in the next paragraph. We stress that these glueing morphisms are only partially defined, on the space of (f_1, f_2) such that the quantity under the integral is integrable on $\mathbb{R}_{>0}$. In that case, we say that f_1 and f_2 are glueable. The glueing morphism corresponding to glueing of several objects along several components is defined in a similar way.

Let us spell out the definition of strict GR in this context. Initial data consist of functions

$$\underline{A} \in \mathcal{C}^0_{\neg\cdot\mathrm{poly}}(\mathbb{R}^3_{>0}), \qquad \underline{B} \in \mathcal{C}^0(\mathbb{R}^3_{>0}), \qquad \underline{C} \in \mathcal{C}^0(\mathbb{R}^3_{>0}), \qquad \underline{D} \in \mathcal{C}^0_{\neg\cdot\mathrm{poly}}(\mathbb{R}_{>0}),$$

with the following symmetries

$$\underline{A}(L_1, L_2, L_3) = \underline{A}(L_1, L_3, L_2), \qquad \underline{C}(L_1, L_2, L_3) = \underline{C}(L_1, L_3, L_2),$$

Definition 9.2 *We say that (A, B, C, D) is admissible with $\neg$-decay for this target theory if there exists $\eta > 0$ such that, for any $s \geq 0$ there exists $r_s > 0$ and $M_s > 0$ such that*

$$\begin{aligned} \sup_{(L_1, L_2, \ell) \in \mathbb{R}^3_{>0}} \frac{\neg(\ell)\ell^{2-\eta+s}\, |\underline{B}(L_1, L_2, \ell)|}{(1+L_1)^{r_s}(1+L_2)^{r_s}\neg(L_1)\neg(L_2)} &\leq M_s, \\ \sup_{(L_1, \ell, \ell') \in \mathbb{R}^3_{>0}} \frac{\neg(\ell)\neg(\ell')(\ell\ell')^{2-\eta+s}\, |\underline{C}(L_1, \ell, \ell')|}{(1+L_1)^{r_s}\neg(L_1)} &\leq M_s. \end{aligned} \tag{49}$$

The recursive formula for the strict GR reads

$$\begin{aligned} &\varpi_{g,n}(L_1, \ldots, L_n) \\ = &\sum_{m=2}^{n} \int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, \underline{B}(L_1, L_m, \ell)\, \varpi_{g,n-1}(\ell, L_2, \ldots, \widehat{L_m}, \ldots, L_n) \\ &+ \frac{1}{2} \int_{\mathbb{R}^2_{>0}} \mathrm{d}\ell \mathrm{d}\ell'\, \ell\, \ell'\, \underline{C}(L_1, \ell, \ell') \Bigg(\varpi_{g-1,n+1}(\ell, \ell', L_2, \ldots, L_n) + \sum_{\substack{J \cup J' = \{L_2, \ldots, L_n\} \\ h+h'=g}} \varpi_{h,1+|J|}(\ell, J)\varpi_{h',1+|J'|}(\ell', J') \Bigg) \end{aligned} \tag{50}$$

with the conventions $\varpi_{0,1} = 0$ and $\varpi_{0,2} = 0$ and where $(L_1, L_2, \ldots, \widehat{L_m}, \ldots, L_n)$ stands for $(L_1, L_2, \ldots, L_n)$ with L_m removed. This formula is just another version of the topological recursion reviewed in Section 7. The initial data plays the role of kernel functions for the integral operators appearing in the recursion.

Lemma 9.3 *If (A, B, C, D) is admissible as in Definition 9.2, then the strict GR is a well-defined functorial assignment $(g, n) \mapsto \varpi_{g,n} \in \mathcal{C}^0_{\neg\cdot\mathrm{poly}}(\mathbb{R}^n_{>0})$.*

Proof. This is proved by induction. Assume the claim has been proved for objects (g', n') such that $2g' - 2 + n' < 2g - 2 + n$. We look at the term $B(L_1, L_m, \ell)\varpi_{g,n-1}(\ell, L_2, \ldots, L_n)$. It is integrable near

$\ell \to \infty$ because the ω factor is $O(\daleth(\ell)\ell^m)$ for some $m > 0$ and we estimate B with the bound (49) for $s = m+1$. It is also integrable near 0, because the ω factor is $O(\daleth(\ell))$, and we estimate B with the bound (49) with $s = 0$. These bounds in fact imply that the outcome of the integration still belongs to $\mathcal{C}^0_{\daleth\cdot\mathrm{poly}}(\mathbb{R}^n_{>0})$. The other terms are treated in the same way. ∎

A priori, the $\varpi_{g,n}$ are symmetric functions of $L_2,\ldots,L_n$ only. Sufficient conditions to ensure that $\varpi_{g,n}$ are symmetric functions of their n variables are provided in Section 6.4. Here they amount to requiring that A is symmetric in its three variables, and for any $L_1, L_2, L_3, L_4 > 0$,

$$\begin{aligned}
\int_{\mathbb{R}_{>0}} \mathrm{d}\ell\,\ell\,\big(\underline{B}(L_1,L_2,\ell)\underline{A}(\ell,L_3,L_4) + \underline{B}(L_1,L_3,\ell)\underline{A}(L_2,\ell,L_4) + \underline{B}(L_1,L_4,\ell)\underline{A}(L_2,\ell,L_3)\big) &= (L_1 \leftrightarrow L_2), \\
\int_{\mathbb{R}_{>0}} \mathrm{d}\ell\,\ell\,\big(\underline{B}(L_1,L_2,\ell)\underline{B}(\ell,L_3,L_4) + \underline{B}(L_1,L_3,\ell)\underline{B}(L_2,\ell,L_4) + \underline{C}(L_1,L_4,\ell)\underline{B}(L_2,\ell,L_3)\big) &= (L_1 \leftrightarrow L_2), \\
\int_{\mathbb{R}_{>0}} \mathrm{d}\ell\,\ell\,\big(\underline{B}(L_1,L_2,\ell)\underline{C}(\ell,L_3,L_4) + \underline{C}(L_1,L_3,\ell)\underline{B}(L_2,\ell,L_4) + \underline{C}(L_1,L_4,\ell)\underline{B}(L_2,\ell,L_3)\big) &= (L_1 \leftrightarrow L_2), \\
\int_{\mathbb{R}_{>0}} \mathrm{d}\ell\,\ell\,\underline{B}(L_1,L_2,\ell)\underline{D}(\ell) + \tfrac{1}{2}\int_{\mathbb{R}^2_{>0}} \mathrm{d}\ell\,\mathrm{d}\ell'\,\underline{C}(L_1,\ell,\ell')\underline{A}(L_2,\ell,\ell') &= (L_1 \leftrightarrow L_2).
\end{aligned}$$

9.5 Integrability with respect to Weil-Petersson volume form

In this paragraph, we work with the target $\mathbf{E}(\mathbf{\Sigma}) = \mathcal{C}^0(\mathcal{T}_\Sigma)$ and the strict target theory $\underline{E}(g,n) = \mathcal{C}^0(\mathbb{R}^n_{>0})$. If $L \in \mathbb{R}^{\pi_0(\partial\Sigma)}_{>0}$, the moduli space $\mathcal{M}_\Sigma(L)$ of bordered Riemann surfaces with hyperbolic boundary lengths L carries the Weil-Petersson volume form ν_Σ. The GR amplitudes $\Omega_{\mathbf{\Sigma}}$ are mapping class group invariant functions on $\mathcal{T}_\Sigma$, therefore descend to continuous functions on the moduli space $\mathcal{M}_\Sigma$ of bordered Riemann surfaces. We are going to show that, under suitable conditions on initial data, the GR amplitudes are integrable with respect to ν_Σ on $\mathcal{M}_\Sigma(L)$, for any $L \in \mathbb{R}^{\pi_0(\partial\Sigma)}$, and their integration over the moduli space are computed by the strict GR.

In the statement below, $\mathbf{\Sigma}_{g,n}$ will be a generic notation for objects in $\mathrm{Bord}_1^\bullet$ which have genus g with n boundaries.

Proposition 9.4 *Let (A,B,C,D) be initial data for $\mathcal{C}^0(\mathcal{T}_\Sigma)$. We assume that, for any $L > 0$, D is integrable over $\mathcal{M}_{\Sigma_{1,1}}(L)$. We also assume that there exists $\eta > 0$ such that for any $s \geq 0$ there exists $r_s > 0$ and $M_s > 0$, such that*

$$\sup_{(L_1,L_2,\ell)\in\mathbb{R}^3_{>0}} \frac{\daleth(\ell)\ell^{2-\eta+s}|B_{\mathbf{\Sigma}_{0,3}}(L_1,L_2,\ell)|}{(1+L_1)^{r_s}(1+L_2)^{r_s}\daleth(L_1)\daleth(L_2)} \leq M_s,$$

$$\sup_{(L_1,\ell,\ell')\in\mathbb{R}^3_{>0}} \frac{\daleth(\ell)\daleth(\ell')(\ell\ell')^{2-\eta+s}\,|C(L_1,\ell,\ell')|}{(1+L_1)^{r_s}\daleth(L_1)} \leq M_s.$$

Then (A,B,C,D) is admissible with $\daleth$-decay (see Definition 4.1), and

$$\left\{\begin{array}{rcl}
\underline{A}(L_1,L_2,L_3) &=& A_{\mathbf{\Sigma}_{0,3}}(L_1,L_2,L_3) \\
\underline{B}(L_1,L_2,L_3) &=& B_{\mathbf{\Sigma}_{0,3}}(L_1,L_2,L_3) \\
\underline{C}(L_1,L_2,L_3) &=& C_{\mathbf{\Sigma}_{0,3}}(L_1,L_2,L_3) \\
\underline{D}(L) &=& \int_{\mathcal{M}_T(L)} \nu_{\Sigma_{1,1}}\, D_{\mathbf{\Sigma}_{1,1}}
\end{array}\right. \tag{51}$$

are admissible initial data with $\daleth$-decay for $\mathcal{C}^0_{\daleth\cdot\mathrm{poly}}(\mathbb{R}^n_{>0})$ (see Definition 9.2). Let Ω and ω be respectively the GR amplitudes for (A,B,C,D), and the strict GR amplitudes for $(\underline{A},\underline{B},\underline{C},\underline{D})$. For any stable (g,n) and $L \in \mathbb{R}^n_{>0}$, we have that

$$\varpi_{g,n}(L) = \int_{\mathcal{M}_{\Sigma_{g,n}}(L)} \Omega_{\mathbf{\Sigma}_{g,n}}\,\nu_{\Sigma_{g,n}}.$$

Proof. The admissibility statement is an easy check from the definitions. So, we can apply Lemma 9.3 and the strict GR amplitudes ω are well-defined.

$\mathcal{T}_{\mathbf{\Sigma}_{0,3}}(L)$ is a point, so the GR amplitude $\Omega_{\mathbf{\Sigma}_{0,3}} := A_{\mathbf{\Sigma}_{0,3}}$ coincides with

$$\varpi_{0,3}(L) := \underline{A}_{\mathbf{\Sigma}_{0,3}}(L) = \int_{\mathcal{T}_{\mathbf{P}}(L)} \Omega_{\mathbf{\Sigma}_{0,3}}\, \nu_{\Sigma_{0,3}}.$$

The definition of $\varpi_{1,1}(L) = \underline{D}(L)$ agrees with the result of integration of $\Omega_{\mathbf{\Sigma}_{1,1}} = D_{\mathbf{T}} = \Omega_{\mathbf{T}}$.

We now assume that $\mathbf{\Sigma}$ is an object in $\mathrm{Bord}_1^{\bullet}$ with Euler characteristic $\chi(\Sigma) \leq -2$. We specialise Proposition 3.9 to the target $\mathcal{C}^0(\mathcal{T}_\Sigma)$, in which the glueing maps are given by (48). Therefore

$$\Omega_{\mathbf{\Sigma}}(\sigma) = \sum_{G \in \mathbb{G}_1^{g,n}} \sum_{\varphi \in \mathbb{M}_G(\mathbf{\Sigma})} \mathbf{E}(\varphi)(\omega_G)(\sigma) \tag{52}$$

where

$$\omega_G(\sigma) = \prod_{i=1}^{2g-2+n} X_{\mathbf{P}_i}\big(\ell_\sigma(\partial P_i)\big). \tag{53}$$

We have explained in Section 3.5 that each term in the sum determines uniquely a homotopy class of pair of pants decomposition $\mathcal{P}$. In this construction, the pair of pants came with an ordering $\mathcal{P} = (P_1, \ldots, P_{2g-2+n})$. We also have a type map $X : [\![1, 2g-2+n]\!] \to \{A, B, C, D\}$. For any given $\mathcal{P}$, we denote $\gamma_1, \ldots, \gamma_{3g-3+n}$ the boundary components of the pair of pants in $\mathcal{P}$, which are interior in Σ. The length and twists of these curves provide Fenchel-Nielsen coordinates on $\mathcal{T}_\Sigma$, in which the measure determined by the Weil-Petersson volume form is $\mathrm{d}\nu_{\Sigma_{g,n}} = 2^{d_{g,n}} \prod_{i=1}^{3g-3+n} \mathrm{d}\ell_i\, \mathrm{d}\theta_i$ where $d_{g,n} := 3g-3+n$. As $(g,n) \neq (1,1)$, the stabiliser of $\mathcal{P}$ in Γ_Σ is generated by the Dehn twists along γ_i, which acts on the Fenchel-Nielsen coordinates via

$$(\ell_j, \theta_j) \to (\ell_j, \theta_j + \delta_{j,i}\ell_i).$$

Therefore, a fundamental domain of $\mathcal{T}_\Sigma(L)$ for the action of $\mathrm{Stab}(\mathcal{P}_G)$ is

$$\big\{(\ell, \theta) \in (\mathbb{R}_{>0} \times \mathbb{R})^{3g-3+n} \quad : \quad 0 \leq \theta_i < \ell_i\big\}.$$

By the integration lemma in [40, Section 8], we have

$$\int_{\mathcal{M}_\Sigma(L)} \Bigg(\sum_{G \in \mathbb{G}_1^{g,n}} \sum_{\varphi \in \mathbb{M}_G(\mathbf{\Sigma})} \big|\mathbf{E}(\varphi)(\omega_G)\big|(\sigma) \Bigg) \mathrm{d}\nu_\Sigma(\sigma) = 2^{d_{g,n}} \sum_{G \in \mathbb{G}_1^{g,n}} \int_{\mathcal{T}_\Sigma(L)/\mathrm{Stab}(\mathcal{P}_G)} \big|\omega_G(\sigma)\big| \prod_{i=1}^{3g-3+n} \mathrm{d}\theta_i\, \mathrm{d}\ell_i$$

where $\mathcal{P}_G$ is the reference homotopy class of pair of pants decomposition corresponding to G. Remark that ω_G only depends on the lengths ℓ_i and the boundary lengths L, so the twists parameters can be integrated out and contribute to a factor $\prod_{i=1}^{3g-3+n} \ell_i$. Besides, each ℓ_i participates at most twice in the factors X_{P_i}. Therefore, the assumptions on (A, B, C, D) guarantees that the integral is finite. By dominated convergence, this shows that $\Omega_{\mathbf{\Sigma}}$ is integrable on $\mathcal{M}_\Sigma(L)$, and applying the integration lemma to (52) yields

$$\int_{\mathcal{M}_\Sigma(L)} \Omega_{\mathbf{\Sigma}}\, \mathrm{d}\nu = 2^{d_{g,n}} \sum_{G \in \mathbb{G}_1^{g,n}} \int_{\mathbb{R}_{>0}^{k(G)}} \prod_{i \in k(G)} \mathrm{d}\ell_i \prod_{i=1}^{2g-2+n} \underline{X}_{P_i}(\ell_{P_i}) \tag{54}$$

where $k(G)$ is the set of curves which are not self-glued in $\mathcal{P}_G$, and $\underline{X}$ is the composition of the type map X with the obvious map $\{A, B, C, D\} \to \{\underline{A}, \underline{B}, \underline{C}, \underline{D}\}$. We have absorbed the extra factors of ℓ_i resulting from twist integrations in the definition of the underlined quantities, see (51), and the remaining integration over lengths is the glueing morphism for the strict target theory described in Section 9.4. The right-hand side of (54) matches with the strict GR amplitudes for the underlined initial data according to Proposition 6.5. ■

9.6 Behaviour at the boundary of Teichmüller space

We consider the behaviour of GR amplitudes in the target theory $\mathbf{E}(\mathbf{\Sigma}) = \mathcal{C}^0(\mathcal{T}_\Sigma)$ at the boundary of the Teichmüller space, *i.e.* when some curves are pinched.

Proposition 9.5 *Let (A,B,C,D) be admissible initial data. Furthermore, we assume that, for any $\varepsilon > 0$, there exists $\varsigma_\varepsilon > 0$ and $M_\varepsilon > 0$ such that*

$$\sup_{\varepsilon \leq L_{b_1}, L_b \leq \varepsilon^{-1}} \sup_{\ell \geq \varepsilon} \left|B_{\mathbf{P}}^b(L_{b_1}, L_b, \ell)\right| e^{\varsigma_\varepsilon \ell} \leq M_\varepsilon, \tag{55}$$

$$\sup_{\varepsilon \leq L_{b_1} \leq \varepsilon^{-1}} \sup_{\ell, \ell' \geq \varepsilon} \left|C_{\mathbf{P}}(L_{b_1}, \ell, \ell')\right| e^{\varsigma_\varepsilon(\ell + \ell')} \leq M_\varepsilon. \tag{56}$$

Let $\mathbf{\Sigma}$ be a stable connected object in $\mathrm{Bord}_1^\bullet$, and μ be disjoint union of finitely many, non-boundary parallel and non-pairwise homotopic simple closed curves in Σ. Denote $\mathfrak{C}(\mathbf{\Sigma}, \mu)$ be the set of c which are homotopy class of embedded pairs of pants P_c such that there exists a connected component μ_i of μ for which $(\partial P_c \smallsetminus \partial\Sigma) \cap \mu_i = \varnothing$.

For any $\varepsilon > 0$, there exists a finite constant $M''_\varepsilon > 0$ such that, for any $\sigma \in \mathcal{T}_\Sigma$ such that the length of all closed curves in Σ except for the connected components of μ is larger than ε, and the length of all boundary components is smaller than ε^{-1}, we have

$$\left|\Omega_{\mathbf{\Sigma}}(\sigma) - \sum_{c \in \mathfrak{C}(\mathbf{\Sigma}, \mu)} \Omega_{\mathbf{\Sigma}}^{(c)}(\sigma)\right| \leq M''_\varepsilon \prod_{\mu_i \in \pi_0(\mu)} \ell_\sigma(\mu_i)^{4\varsigma_\varepsilon - 1 - \varepsilon}$$

where $\Omega_{\mathbf{\Sigma}}^{(c)}$ is the term in the GR formula (46) corresponding to c.

Proof. We detail the proof when μ is a simple closed curve in the interior of Σ, as the case of several components is similar. We denote $\mathfrak{C}'(\mathbf{\Sigma}, \mu)$ the complement of $\mathfrak{C}(\mathbf{\Sigma}, \mu)$ in $\mathfrak{C}(\mathbf{\Sigma})$. For any $c \in \mathfrak{C}'(\mathbf{\Sigma}, \mu)$, the (multi)curve γ_c intersects μ. As μ must at least enter and exit P_c, it must intersects at least twice γ_c. By the collar Lemma 8.1, we must have

$$\ell_\sigma(\gamma_c) \geq 2 w_\sigma(\mu) \tag{57}$$

where $w_\sigma(\mu) > 0$ is the increasing function of $\ell_\sigma(\mu)$ defined by the equation

$$\sinh\Big(\frac{w_\sigma(\mu)}{2}\Big)\sinh\Big(\frac{\ell_\sigma(\mu)}{2}\Big) = 1.$$

When $\ell_\sigma(\mu) \to 0$, we have that

$$w_\sigma(\mu) \sim 2\ln\Big(\frac{4}{\ell_\sigma(\mu)}\Big) \tag{58}$$

and the left-hand side is always larger than the right-hand side.

Let $\varepsilon > 0$, and $\sigma \in \mathcal{T}_\Sigma$ such that the only simple closed curve with length ε is μ. In particular, for any $c \in \mathfrak{C}'(\mathbf{\Sigma}, \mu)$, we have $\mathrm{sys}_\sigma(\Sigma_c) \geq \varepsilon$ and $\ell_\sigma(\beta) \geq \varepsilon$ for any $\beta \in \pi_0(\partial P_c)$. Let us also assume that each boundary component of Σ has length $\leq \varepsilon^{-1}$. We obtain the upper bound

$$\left|\sum_{c \in \mathfrak{C}'(\mathbf{\Sigma}, \mu)} X_{\mathbf{P}_c}(\ell_\sigma(P_c))\, \Omega_{\mathbf{\Sigma}_c}(\sigma|_{\Sigma_c})\right| \leq \sum_{c \in \mathfrak{C}'(\mathbf{\Sigma}, \mu)} M_\varepsilon\, e^{-\varsigma_\varepsilon \ell_\sigma(\gamma_c)} \|\Omega_{\mathbf{\Sigma}_c}\|_\varepsilon$$

where X is either a B or a C as in the GR formula, depending on the topology of $\Sigma - P_c$. By construction of GR, $\Omega_{\mathbf{\Sigma}_c}$ is a functorial assignment and has finite $\|\cdot\|_\varepsilon$ which only depends on the topology of Σ_c. Since there are finitely many possible topologies, there exists a finite constant M'_ε such that

$$\forall c \in \mathfrak{C}'(\mathbf{\Sigma}, \mu), \qquad \|\Omega_{\mathbf{\Sigma}_c}\|_\varepsilon \leq M'_\varepsilon.$$

Using the assumption (55)-(56) and the lower bound (57), we then obtain

$$\left| \sum_{c \in \mathfrak{C}'(\mathbf{\Sigma}, \mu)} X_{\mathbf{P}_c}(\ell_\sigma(P_c))\, \Omega_{\mathbf{\Sigma}_c}(\sigma|_{\Sigma_c}) \right| \leq M_\varepsilon\, M'_\varepsilon \sum_{\substack{\gamma \in S_\Sigma^\circ \\ \ell_\sigma(\gamma) \geq 2 w_\sigma(\mu)}} e^{-\varsigma_\varepsilon\, \ell_\sigma(\gamma)}.$$

We estimate it as follows

$$\begin{aligned} \sum_{\substack{\gamma \in S_\Sigma^\circ \\ \ell_\sigma(\gamma) \geq 2 w_\sigma(\Gamma)}} e^{-\varsigma_\varepsilon \ell_\sigma(\gamma)} & \leq \sum_{n \geq 0} \big(\mathrm{N}_\sigma(2 w_\sigma(\mu) + n) - \mathrm{N}_\sigma(2 w_\sigma(\mu) + n + 1) \big)\, e^{-\varsigma_\varepsilon (2 w_\sigma(\mu) + n)} \\ & \leq \mathrm{N}_\sigma(2 w_\sigma(\mu))\, e^{-2 w_\sigma(\mu)} \end{aligned}$$

where $\mathrm{N}_\sigma(\ell)$ is the number of multicurves with length bounded above by ℓ. Using Mirzakhani's fine upper bound in Theorem 8.3, we deduce

$$\left| \sum_{c \in \mathfrak{C}'(\mathbf{\Sigma}, \mu)} X_{\mathbf{P}_c}(\ell_\sigma(P_c))\, \Omega_{\mathbf{\Sigma}_c}(\sigma|_{\Sigma_c}) \right| \leq M''_\varepsilon \ell_\sigma(\mu)^{4\varsigma_\varepsilon - 1} \left(\ln\Big(\frac{4}{\ell_\sigma(\mu)} \Big) \right)^{6g-6+2n}$$

where g and n is the genus and number of boundary components of $\mathbf{\Sigma}$ and M''_ε is some finite constant. We can get rid of the logarithm factor by replacing ς_ε with a slightly smaller constant. ■

In particular, when $\varsigma_\varepsilon > \frac{1}{4}$, the contribution in $\Omega_{\mathbf{\Sigma}}$ of pair of pants intersecting the components μ_i of μ vanishes when $\ell_\sigma(\mu_i) \to 0$. If the initial data had a stronger decay like e^{-l^α} for some $\alpha > 1$ with respect to the length l of the curve we glue on, the same proof would conclude that the contribution in $\Omega_{\mathbf{\Sigma}}$ of the pair of pants intersecting μ_i decays faster than any polynomial when $\ell_\sigma(\mu_i) \to 0$, *i.e.* would be negligible compared to Taylor series expansion (if it exists) at any order in this length variable.

10 Revisiting Mirzakhani-McShane identities

In this section, we review the famous generalisation [40] of McShane identity [39], which is the prototype of the geometric recursion. It will also provide us with weaker admissibility conditions for initial data. By symmetry considerations, we also conjecture a new McShane-type identity for the sphere with four boundaries. Finally, we exploit the Mirzakhani-McShane identity to give a systematic lift of the strict GR to GR.

10.1 The identity and its consequences

Let us work in the target theory $\mathbf{E}(\mathbf{\Sigma}) = \mathcal{C}^0(\mathcal{T}_\Sigma)$ for $\mathrm{Bord}_1^\bullet$, and consider what we will call Mirzakhani initial data,

$$\begin{cases} \underline{\mathsf{A}}(L_1, L_2, L_3) & = & 1 \\ \underline{\mathsf{B}}(L_1, L_2, \ell) & = & 1 - \frac{1}{L_1} \ln\left(\frac{\cosh\left(\frac{L_2}{2}\right) + \cosh\left(\frac{L_1 + \ell}{2}\right)}{\cosh\left(\frac{L_2}{2}\right) + \cosh\left(\frac{L_1 - \ell}{2}\right)} \right) \\ \mathsf{C}(L_1, \ell, \ell') & = & \frac{1}{L_1} \ln\left(\frac{\exp\left(\frac{L_1}{2}\right) + \exp\left(\frac{\ell + \ell'}{2}\right)}{\exp\left(-\frac{L_1}{2}\right) + \exp\left(\frac{\ell + \ell'}{2}\right)} \right) \\ \underline{\mathsf{D}}(L) & = & \frac{L^2}{24} + \frac{\pi^2}{6} \end{cases}. \tag{59}$$

This is an initial data for strict GR, which determines the following initial data for GR

$$\begin{aligned} \mathsf{X}_{\mathbf{P}}(\sigma) &= \underline{X}(\ell_\sigma(\partial P)), \qquad \mathsf{X} \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}, \\ \mathsf{D}_{\mathbf{T}}(\sigma) &= \sum_{\gamma \in \hat{S}_T^\circ} \mathsf{C}_{\mathbf{P}_\gamma}(\sigma|_{P_\gamma}). \end{aligned}$$

Let us introduce the function

$$\mathsf{F}(x) = 2\ln(1 + e^{-\frac{x}{2}})$$

which has the property $\mathsf{F}(x) - \mathsf{F}(-x) = -x$. This initial data can also be decomposed as

$$\underline{\mathsf{B}}(L_1, L_2, \ell) = \frac{1}{2L_1}\Big(\mathsf{F}(\ell + L_2 - L_1) + \mathsf{F}(\ell - L_2 - L_1) - \mathsf{F}(\ell - L_2 + L_1) - \mathsf{F}(\ell + L_2 + L_1)\Big), \tag{60}$$

$$\underline{\mathsf{C}}(L_1, \ell, \ell') = \frac{1}{L_1}\Big(\mathsf{F}(\ell + \ell' - L_1) - \mathsf{F}(\ell + \ell' + L_1)\Big). \tag{61}$$

The formula for the GR amplitudes would read

$$\begin{aligned} \Omega_{\mathbf{\Sigma}}(\sigma) = & \sum_{b \in \pi_0(\partial_+\Sigma)} \sum_{c \in \mathfrak{C}_{b_1,b}(\Sigma)} \underline{\mathsf{B}}(\ell_\sigma(b_1), \ell_\sigma(b), \ell_\sigma(\gamma_c))\, \Omega_{\mathbf{\Sigma}_c}(\sigma|_{\Sigma_c}) \\ & + \frac{1}{2} \sum_{c \in \mathfrak{C}_{b_1,b_1}(\Sigma)} \underline{\mathsf{C}}(\ell_\sigma(b_1), \ell_\sigma(\gamma_c^1), \ell_\sigma(\gamma_c^2))\, \Omega_{\mathbf{\Sigma}_c}(\sigma|_{\Sigma_c}), \end{aligned} \tag{62}$$

where b_1 is the negatively oriented boundary of Σ.

However, the above B and C decay exponentially in the lengths ℓ and ℓ' for fixed L_1 and L_2, but we can only say that it is uniformly bounded when $L_1, L_2, \ell, \ell' \to \infty$. Therefore, this initial data is not admissible in the sense of Definition 3.5, and we return to this question in the next paragraph.

The Mirzakhani-McShane identity nevertheless states that

Theorem 10.1 *[40] For any stable $\mathbf{\Sigma}$, $\Omega_{\mathbf{\Sigma}}$ is the constant function 1 on $\mathcal{T}_\Sigma$.*

There is a similar identity for tori with one boundary [40, Equation 1.4]. With our notations it reads

$$\forall \sigma \in \mathcal{T}_T, \qquad \mathsf{D}_{\mathbf{T}}(\sigma) = 1 = \sum_{\gamma \in \hat{S}_T^\circ} \underline{\mathsf{C}}(\ell_\sigma(\partial T), \ell_\sigma(\gamma), \ell_\sigma(\gamma)) \tag{63}$$

where $\mathbf{T}$ is an object in $\mathrm{Bord}_1^\bullet$ which is a torus with one boundary. We see that this choice of $\mathsf{D}_{\mathbf{T}}$ coincides with the choice described in Lemma 4.3.

In [40], Mirzakhani used these identities to prove a recursive formula for the Weil-Petersson volumes of the moduli space of bordered surfaces

$$\mathsf{V}_{g,n}(L) = \int_{\mathcal{M}_\Sigma(L)} 1 \cdot \nu_\Sigma.$$

In fact, $\mathsf{V}_{g,n}(L)$ are the strict GR amplitudes for the initial data (59) – and the value of $\underline{D}(L)$ was chosen to be equal to the known value of $\mathsf{V}_{1,1}(L)$.

We also know [24] that an equivalent form of this recursion, after Laplace transform in the boundary lengths, is the topological recursion of Eynard and Orantin (see Section 7.3), for the initial data

$$x(z) = z^2, \qquad \omega_{0,1}(z) = \frac{\sin(2\pi z)}{2\pi}\, z\mathrm{d}z, \qquad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}.$$

More generally, in Section 11 we will investigate more generally the formulation of (strict) GR in terms of the Laplace transform of the boundary lengths and its relation to the original topological recursion.

We now review the relation between the Weil-Petersson volumes $\mathsf{V}_{g,n}(L)$ and intersection theory on Deligne-Mumford compactification of the moduli space $\overline{\mathcal{M}}_{g,n}$ of genus g Riemann surfaces with n punctures $[S, p_1, \ldots, p_n]$. We recall that $\overline{\mathcal{M}}_{g,n}$ is a complex orbifold for which Poincaré duality holds. Let us denote ψ_i the first Chern class of the cotangent line bundle $T^*_{p_i}S$ at the i-th puncture. We also denote κ_d the class of complex degree d obtained by pushforward of ψ_{n+1}^{d+1} via the morphism $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ forgetting the last puncture. It is well-known that the cohomology class of the Weil-Petersson symplectic form on $\overline{\mathcal{M}}_{g,n}$ is $2\pi^2\kappa_1$ [56]. By examination of the symplectic reduction on the space $\mathcal{M}_{\boldsymbol{\Sigma}_{g,n}}$ with the moment map $(\frac{L_1^2}{2}, \ldots, \frac{L_n^2}{2})$, Mirzakhani proved

Theorem 10.2 *[41] For $2g - 2 + n > 0$, we have*

$$\mathsf{V}_{g,n}(L) = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(2\pi^2\kappa_1 + \sum_{i=1}^{n} \frac{L_i^2}{2}\,\psi_i\right)$$

where the exponential is understood by expanding in Taylor series, and keeping only the terms of complex cohomology degree $d_{g,n} = 3g - 3 + n$.

10.2 Another admissibility condition

Non-admissibility of Mirzakhani initial data

As already mentioned, Mirzakhani initial data is not admissible in the sense of Definition 3.5. It is rather a limit of admissible initial data.

Lemma 10.3 *For any $s > 1$, we have*

$$\sup_{L_1, L_2, \ell \geq \varepsilon} \underline{\mathsf{B}}(L_1, L_2, \ell)\,\ell^s = +\infty\,, \qquad \sup_{L_1, \ell, \ell' \geq \varepsilon} \underline{\mathsf{C}}(L_1, \ell, \ell')\,(\ell + \ell')^s = +\infty\,.$$

Proof. The estimate

$$\underline{\mathsf{C}}(\ell + \ell', \ell, \ell') = \frac{2}{\ell + \ell'}\,\ln\left(\frac{2}{1 + e^{-\ell - \ell'}}\right) \sim \frac{2\ln 2}{\ell + \ell'}$$

when $\ell + \ell' \to \infty$ implies the claim for $\underline{\mathsf{C}}$. The analysis of $\underline{\mathsf{B}}(\ell, 2\ell, \ell)$ when $\ell \to \infty$ gives the claim for $\underline{\mathsf{B}}$. ∎

Lemma 10.4 *For $\alpha \in (0,1)$, let $(\mathsf{A}^{(\alpha)}, \mathsf{B}^{(\alpha)}, \mathsf{C}^{(\alpha)})$ be the initial data obtained from (59) by replacing L_1 with αL_1, and induce $\mathsf{D}^{(\alpha)}$ from (19). This initial data is admissible with $\neg$-decay (see Definition 4.1) for the choice $\neg(\ell) = e^{\frac{\alpha\ell}{2}}$.*

Proof. For any $x \in \mathbb{R}$, we have $0 \leq \mathsf{F}(x) \leq 2\,e^{-\frac{x}{2}}$. We deduce

$$\frac{\neg(\ell)\neg(\ell')}{\neg(L_1)}\,\underline{\mathsf{C}}^{(\alpha)}(L_1, \ell, \ell') \leq \frac{2\,\neg(\ell)\neg(\ell')}{\neg(L_1)\,\alpha L_1}\,e^{\frac{\alpha L_1 - \ell - \ell'}{2}} = \frac{2\,e^{-\frac{1-\alpha}{2}(\ell + \ell')}}{\alpha L_1}$$

and therefore, for any $\varepsilon > 0$, any $s > 0$ and $\alpha \in [\varepsilon, 1)$ we have

$$\sup_{L_1, \ell, \ell' \geq \varepsilon} \frac{\neg(\ell)\neg(\ell')}{\neg(L_1)}\,\underline{\mathsf{C}}^{(\alpha)}(L_1, \ell, \ell')\,(\ell + \ell')^s < +\infty.$$

Likewise, we bound

$$\frac{\neg(\ell)}{\neg(L_1)\neg(L_2)}\,\underline{\mathsf{B}}^{(\alpha)}(L_1, L_2, \ell) \leq \frac{\neg(\ell)}{\neg(L_1)\neg(L_2)}\,\frac{e^{\frac{\alpha L_1 - L_2 - \ell}{2}} + e^{\frac{\alpha L_1 + L_2 - \ell}{2}}}{\alpha L_1} = \frac{e^{-\frac{1-\alpha}{2}\ell - \frac{1+\alpha}{2}L_2} + e^{-\frac{1-\alpha}{2}(\ell + L_2)}}{\alpha L_1}$$

which implies for any $s>0$

$$\sup_{L_1,L_2,\ell\geq\varepsilon}\frac{\daleth(\ell)}{\daleth(L_1)\daleth(L_2)}\,\underline{\mathsf{B}}^{(\alpha)}(L_1,L_2,\ell)\,\ell^s<+\infty.$$

■

Yet, the Mirzakhani-McShane identities tells us that $\Omega_{\mathbf{\Sigma}}$ is well-defined, *i.e.* the series in (62) are absolutely convergent on any compact, and the result actually belongs to the space $E'(\mathbf{\Sigma})$ of continuous functions which are bounded on the thick part of the Teichmüller space.

Therefore, we may circumvent the non-admissibility of $(\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D})$ by running GR for the initial data depending on $\alpha\in(0,1)$ to get functorial functions $\Omega^{(\alpha)}_{\mathbf{\Sigma}}$ for any stable $\mathbf{\Sigma}$, and then take the limit $\alpha\to 1$ which reaches the function 1 on $\mathcal{C}^0(\mathcal{T}_\Sigma)$ according to the Mirzakhani-McShane identities.

Note that the properties of GR with $\daleth$-decay a priori show that for any $\alpha<1$, $\Omega^{(\alpha)}_{\mathbf{\Sigma}}$ can grow at most like $O(e^{\alpha\ell_\sigma(\partial\Sigma)})$ in the boundary lengths in the thick part of the Teichmüller space. But, as $\Omega^{(\alpha)}_{\mathbf{\Sigma}}$ is an increasing function of α, we in fact know that

$$\Omega^{(\alpha)}_{\mathbf{\Sigma}}(\sigma)\leq\Omega_{\mathbf{\Sigma}}(\sigma)=1.$$

This uniform bound clearly cannot be reached with the previous results on GR.

M-admissibility and GR

At this stage, we find instructive to turn the point of view upside down: we will formulate weaker admissibility conditions using the bounds implied by the Mirzakhani-McShane identities, under which the main Theorem 3.7 of definition of the GR amplitudes will continue to hold.

We should first extract the relevant property of the hyperbolic length spectrum which Theorem 10.1 provides.

Corollary 10.5 *For any $\varepsilon>0$, there exists $M(\varepsilon,g,n)>0$ such that, for any stable Σ of genus g with n boundaries, we have*

$$\sup_{\sigma\in K_\Sigma(\varepsilon)}\frac{1}{\ell_\sigma(b_1)\ell_\sigma(b)}\sum_{c\in\mathfrak{C}_{b_1,b}(\Sigma)}\ln\big(1+e^{\frac{\ell_\sigma(b_1)+\ell_\sigma(b)-\ell_\sigma(\gamma_c)}{2}}\big)\leq M(\varepsilon,g,n),\tag{64}$$

$$\sup_{\sigma\in K_\Sigma(\varepsilon)}\frac{1}{\ell_\sigma(b_1)}\sum_{c\in\mathfrak{C}_{b_1,b_1}(\Sigma)}\ln\big(1+e^{\frac{\ell_\sigma(b_1)-\ell_\sigma(\gamma_c)}{2}}\big)\leq M(\varepsilon,g,n).\tag{65}$$

We cannot prove this result by zeta function arguments using the estimate of Theorem 8.3 on the number of simple closed curves of bounded length on bordered surfaces. In fact, in the proof of Theorem 8.3, we have controlled the length spectra on bordered surfaces uniformly in terms of the length spectra on punctured surfaces. In doing so, we have lost the dependence of N_σ in the boundary lengths, and it could explain why we are unable to give a direct proof of Corollary 10.5.

Proof. As B and C are non-negative, Theorem 10.1 implies that

$$\sup_{\sigma\in K_\Sigma(\varepsilon)}\sum_{c\in\mathfrak{C}_{b_1,b}(\Sigma)}\underline{\mathsf{B}}(\ell_\sigma(b_1),\ell_\sigma(b),\ell_\sigma(\gamma_c))\quad\leq\quad 1,\tag{66}$$

$$\sup_{\sigma\in K_\Sigma(\varepsilon)}\sum_{c\in\mathfrak{C}_{b_1,b_1}(\Sigma)}\underline{\mathsf{C}}(\ell_\sigma(b_1),\ell_\sigma(\gamma_c^1),\ell_\sigma(\gamma_c^2))\quad\leq\quad 1.\tag{67}$$

We decompose

$$\underline{\mathsf{C}}(L_1,\ell,\ell')=\frac{2}{L_1}\ln(1+e^{\frac{L_1-\ell-\ell'}{2}})-\frac{2}{L_1}\ln(1+e^{-\frac{L_1+\ell+\ell'}{2}}).$$

The second term satisfies the condition (6) in our decay axiom. In particular, the proof of our Theorem 3.7 *via* the uniform convergence of the zeta function implies that there exists a constant $M'(\varepsilon, g, n) > 0$ such that

$$\sup_{\sigma \in K_\Sigma(\varepsilon)} \sum_{c \in \mathfrak{C}_{b_1,b_1}(\Sigma)} \frac{1}{\ell_\sigma(b_1)} \ln(1 + e^{\frac{\ell_\sigma(b_1) - \ell_\sigma(\gamma_c^1) - \ell_\sigma(\gamma_c^2)}{2}}) \leq M'(\varepsilon, g, n),$$

hence the claimed estimate (65).

For the second estimate, we use the decomposition (60) of B, in the following form

$$\begin{aligned} \underline{B}(L_1, L_2, \ell) + \frac{\mathsf{F}(\ell + L_2 + L_1)}{2L_1} &= \frac{\mathsf{F}(\ell + L_2 - L_1)}{2L_1} + \frac{\mathsf{F}(\ell - L_2 - L_1) - \mathsf{F}(\ell - L_2 + L_1)}{2L_1} \\ &\geq \frac{\mathsf{F}(\ell + L_2 - L_1)}{2L_1} \end{aligned} \tag{68}$$

as $\mathsf{F}(x)$ decreases with x. We then replace $\ell = \ell_\sigma(\gamma_c)$ and sum over $c \in \mathfrak{C}_{b_1,b}(\Sigma)$ for a $\sigma \in K_\Sigma(\varepsilon)$. The contribution of the first term in the left-hand side of (68) is bounded due to (66), the second term satisfies the decay axiom (5) therefore also gives a bounded contribution. We conclude that there exists a constant $M''(\varepsilon, g, n) > 0$ such that

$$\sup_{\sigma \in K_\Sigma(\varepsilon)} \sum_{c \in \mathfrak{C}_{b_1,b}(\Sigma)} \frac{\mathsf{F}(\ell_\sigma(\gamma_c) + \ell_\sigma(b) - \ell_\sigma(b_1))}{\ell_\sigma(b_1)} \leq M''(\varepsilon, g, n).$$

As b_1 and b play a symmetric role in $\mathfrak{C}_{b_1,b}(\Sigma)$, and $K_\Sigma(\varepsilon)$ is invariant under braiding of b_1 and b, this also implies

$$\sup_{\sigma \in K_\Sigma(\varepsilon)} \sum_{c \in \mathfrak{C}_{b_1,b}(\Sigma)} \frac{\mathsf{F}(\ell_\sigma(\gamma_c) + \ell_\sigma(b_1) - \ell_\sigma(b))}{\ell_\sigma(b)} \leq M''(\varepsilon, g, n).$$

We can now rewrite

$$\begin{aligned} \frac{1}{L_2} \underline{B}(L_1, L_2, \ell) + \frac{\mathsf{F}(\ell + L_2 + L_1)}{2L_1L_2} + \frac{\mathsf{F}(\ell - L_2 + L_1)}{2L_1L_2} &= \frac{\mathsf{F}(\ell + L_2 - L_1)}{2L_1L_2} + \frac{\mathsf{F}(\ell - L_2 - L_1)}{2L_1L_2} \\ &\geq \frac{\mathsf{F}(\ell - L_2 - L_1)}{2L_1L_2} \end{aligned}$$

because F is nonnegative. We substitute $(L_1, L_2, \ell) = (\ell_\sigma(b_1), \ell_\sigma(b), \ell_\sigma(\gamma_c))$ and since $\ell_\sigma(b)$ is bounded from below by ε for $\sigma \in K_\Sigma(\varepsilon)$, we deduce from the previous bounds there exists a constant $M'''(\varepsilon, g, n) > 0$ such that

$$\sup_{\sigma \in K_\Sigma(\varepsilon)} \sum_{c \in \mathfrak{C}_{b_1,b}(\Sigma)} \frac{\mathsf{F}(\ell_\sigma(\gamma_c) - \ell_\sigma(b) - \ell_\sigma(b_1))}{\ell_\sigma(b_1)\ell_\sigma(b)} \leq M'''(\varepsilon, g, n)$$

which is the claim (64). ■

Definition 10.6 *An initial data (A, B, C, D) for the target theory $\mathcal{C}^0(\mathcal{T}_\Sigma)$ is M-admissible if for any $\varepsilon > 0$, there exists a constant $M(\varepsilon) > 0$ such that, for any $L_1, L_2, \ell, \ell' \geq \varepsilon$ we have*

$$\begin{aligned} \left|\underline{B}(L_1, L_2, \ell)\right| &\leq \frac{M(\varepsilon) \ln(1 + e^{\frac{L_1 + L_2 - \ell}{2}})}{L_1L_2}, & (69) \\ \left|\underline{C}(L_1, L_2, \ell)\right| &\leq \frac{M(\varepsilon) \ln(1 + e^{\frac{L_1 - \ell - \ell'}{2}})}{L_1}. & (70) \end{aligned}$$

We say that (A, B, C, D) is M-bounded if the constant $M(\varepsilon)$ above can be chosen independent of ε.

Proposition 10.7 *Let (A,B,C,D) be an M-admissible initial data. Then, for any stable $\mathbf{\Sigma}$, the GR amplitude $\Omega_{\mathbf{\Sigma}}$ is well-defined. More precisely, the series in (46) converges uniformly on any compact, its limit is an element of $E'(\mathbf{\Sigma})$, and it is functorial. In particular, $\Omega_{\mathbf{\Sigma}}$ is mapping class group invariant.*

If (A,B,C,D) is M-bounded, then $\Omega_{\mathbf{\Sigma}}$ is a function on $\mathcal{T}_\Sigma$ bounded by the constant M. A fortiori, $\Omega_{\mathbf{\Sigma}}$ is integrable with respect to the Weil-Petersson volume form, and

$$\varpi_{g,n}(L) = \int_{\mathcal{M}_{\Sigma_{g,n}}(L)} \Omega_{\mathbf{\Sigma}_{g,n}} \nu_{\Sigma_{g,n}}$$

belongs to $\mathcal{C}^0_{\mathrm{poly}}(\mathbb{R}^n_{>0})$ and are the strict GR amplitudes for the initial data $(\underline{A},\underline{B},\underline{C},\underline{D})$ which is related to (A,B,C,D) via (51).

Proof. To prove the first part, we repeat the proof of Theorem 3.7, but use the bounds of Corollary 10.5 instead of (9). In the second part, the assumption and the monotonicity of F implies the existence of a universal constant $M > 0$ such that, for any $L_1, L_2, \ell, \ell' > 0$ we can control our initial data by Mirzakhani initial data

$$\left|\underline{B}(L_1,L_2,\ell)\right| \le M\, \underline{\mathsf{B}}(L_1,L_2,\ell)\,, \qquad \left|\underline{C}(L_1,\ell,\ell')\right| \le M\, \underline{\mathsf{C}}(L_1,\ell,\ell').$$

Therefore

$$\forall \sigma \in \mathcal{T}_\Sigma, \qquad \Omega_{\mathbf{\Sigma}}(\sigma) \le M\, \Omega_{\mathbf{\Sigma}}(\sigma) = M.$$

We can then follow the proof of Proposition 9.4 to check that integration over the moduli space followed by strict GR coincide with GR followed by integration over the moduli space. ∎

COMMENTS.

In case we wish to induce initial data for tori with one boundary from B or C, the result of Lemma 4.3 also easily extends to M-admissible/M-bounded (A,B,C).

We remark that M-admissibility implies that B and C decay like $O(e^{-\frac{\ell_\sigma(\gamma)}{2}})$ with respect to the length of the curve we glue on provided the boundary lengths L_i remain bounded. Such a decay is not implied by the admissibility condition in Definition 3.5, therefore M-admissibility is neither weaker nor stronger than admissibility.

In the same target $\mathcal{C}^0(\mathcal{T}_\Sigma)$, one could formulate other notions of admissibility as soon as one has of some B^{ref} and C^{ref} such that, for any stable $\mathbf{\Sigma}$ and any $\varepsilon > 0$, there exist constants $M(\varepsilon,g,n) > 0$ such that

$$\begin{aligned}
\sum_{\sigma \in K_\Sigma(\varepsilon)} \sum_{c \in \mathfrak{C}_{b_1,b}(\Sigma)} \left|B^{b,\mathrm{ref}}_{\mathbf{P}_c}(\sigma|_{\mathbf{P}_c})\right| &\le M(\varepsilon,g,n),\\
\sum_{\sigma \in K_\Sigma(\varepsilon)} \sum_{c \in \mathfrak{C}_{b_1,b_1}(\Sigma)} \left|C^{\mathrm{ref}}_{\mathbf{P}_c}(\sigma|_{\mathbf{P}_c})\right| &\le M(\varepsilon,g,n).
\end{aligned}$$

In particular, we stress that such estimates are uniform when the boundary lengths become large owing to the definition of $K_\Sigma(\varepsilon)$ for bordered surfaces Σ.

M-admissibility only makes sense in target theories involving functions over the Teichmüller space, as it relies on properties of the hyperbolic length spectrum like Corollary 10.5. In an arbitrary target, one may add assumptions on the property of the length functions provided with the target to obtain weaker condition of convergence of the series involved in the GR formula. The value of such additional assumptions should be put in balance with the amount of properties of lengths functions the readers want to establish in the specific target theories they would like to work with.

10.3 Fibering over Teichmüller space

Imagine that $\mathbf{F}$ is a target theory maybe without the data of length functions. We may call this a pre-target theory. The typical case we have in mind are functors coming from topological field theories or conformal field theories (see Section 13). We may construct a new target theory by considering continuous functions from Teichmüller space with values in $\mathbf{F}$,

$$\mathbf{E}(\Sigma) = \mathcal{C}^0(\mathcal{T}_\Sigma, \mathbf{F}(\mathbf{\Sigma})). \tag{71}$$

We define the union and glueing morphisms for $\mathbf{E}$ by combining those of $\mathbf{F}$ and those of Sections 9.1-9.2. We equip $\mathbf{E}(\Sigma)$ with the seminorms indexed by $i \in \mathscr{I}_{\mathbf{\Sigma}}^{\mathbf{F}}$, $\alpha \in \mathscr{A}_{\mathbf{\Sigma}}^{(i),\mathbf{F}}$, $\varepsilon > 0$ and F a compact subset of $K_\Sigma(\varepsilon)$,

$$|f|_{\alpha,F} = \sup_{\sigma \in F} |f(\sigma)|_\alpha.$$

We can equip $\mathbf{E}$ with the collection of length functions induced by hyperbolic lengths as in (43)

$$\ell_{\alpha,K} = \min_{\sigma \in F} \ell_\sigma(\gamma)$$

which in fact does not depend on α. On top of the notion of admissibility of $\mathbf{E}$-valued initial data from Definition 3.5, one can also formulate the notion of M-admissibility (and M-boundedness), by replacing in (69)-(70) the absolute value with the norms the space $\mathbf{F}(\mathbf{\Sigma})$ is functorially equipped with. For M-admissible initial data, the result of Theorem 3.7 still holds, *i.e.* the $\mathbf{E}$-valued GR amplitudes are well-defined and functorial.

If (A, B, C, D) are initial data for $\mathbf{F}$, we can introduce canonical initial data $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ for $\mathbf{E}$ with the formulae

$$\begin{cases} \tilde{A}_{\mathbf{P}}(L_1, L_2, L_3) & = & 1 \\ \tilde{B}_{\mathbf{P}}(L_1, L_2, \ell) & = & \left\{1 - \frac{1}{L_1} \ln\left(\frac{\cosh(\frac{L_2}{2}) + \cosh(\frac{L_1+\ell}{2})}{\cosh(\frac{L_2}{2}) + \cosh(\frac{L_1-\ell}{2})}\right)\right\} \cdot B_{\mathbf{P}} \\ \tilde{C}_{\mathbf{P}}(L_1, \ell, \ell') & = & \frac{1}{L_1} \ln\left(\frac{e^{\frac{L_1}{2}} + e^{\frac{\ell+\ell'}{2}}}{e^{-\frac{L_1}{2}} + e^{\frac{\ell+\ell'}{2}}}\right) \cdot C_{\mathbf{P}} \\ \tilde{D}_{\mathbf{T}}(\sigma) & = & D_{\mathbf{T}}\,. \end{cases} \tag{72}$$

Obviously, $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is always M-bounded.

If the mapping class groups act trivially on $\mathbf{F}(\mathbf{\Sigma})$, using the Mirzakhani-McShane identity, we deduce that the GR amplitudes for $\mathbf{E}$ attached to (72) will be constant functions $\Omega_{\mathbf{\Sigma}}$ on Teichmüller space. This constant value in $\mathbf{F}(\Sigma)$ can be considered as a definition of the GR amplitude for $\mathbf{F}$ with initial data (A, B, C, D). This trick circumvents the potential absence of length functions in certain would-be target theories.

If the mapping class group act non-trivially on $\mathbf{F}(\mathbf{\Sigma})$, this argument does not apply and the GR amplitudes attached to (72) could *a priori* be interesting functions on the Teichmüller space.

10.4 Symmetry issues

$\mathcal{C}^0(\mathcal{T}_\Sigma)$ can also be considered as a symmetric target theory. We see that for the Mirzakhani initial data, Ω_Σ is invariant under braidings of all boundaries because it is the function 1. Without using the full knowledge of Theorem 10.1, it would be interesting to understand why the result of GR is symmetric.

It is possible that it occurs because $(\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D})$ actually induces an initial data for symmetric GR, *i.e.* it satisfies the relations described in Section 5.1. Let us make our suspicions more precise.

A is obviously invariant under all braidings. Since the BA and D relation are respectively equivalent to the invariance of $\Omega_{\Sigma_{0,4}}$ and $\Omega_{\Sigma_{1,2}}$ under all braidings, so they hold for Mirzakhani initial data. The BB-AC and BC relations are used in proving invariance under all braidings in symmetric GR, but from the proof of Theorem 5.3 one cannot conclude it is equivalent to this invariance for some Ω_Σ. We thus ask the following.

Question 10.8 *Do the Mirzakhani initial data satisfies the BC and BB-CA relation? Namely, if* $\mathbf{\Sigma}$ *is a sphere with four boundaries* (b_1,b_2,b_3,b_4)*, let us define the following functions of* $\sigma \in \mathcal{T}_\Sigma$

$$
\begin{aligned}
\mathsf{Q}(\sigma) &= \sum_{c\in\mathfrak{C}^{b_3}_{b_1,b_1}(\mathbf{\Sigma})} \mathsf{C}(L_1,L_3,\ell_\sigma(\gamma_c))\mathsf{B}(L_2,\ell_\sigma(\gamma_c),L_4) + \sum_{c\in\mathfrak{C}^{b_4}_{b_1,b_1}(\mathbf{\Sigma})} \mathsf{C}(L_1,L_4,\ell_\sigma(\gamma_c))\mathsf{B}(L_2,\ell_\sigma(\gamma_c),L_3) \\
&\quad + \sum_{c\in\mathfrak{C}_{b_1,b_2}(\mathbf{\Sigma})} \mathsf{B}(L_1,L_2,\ell_\sigma(\gamma_c))\mathsf{C}(\ell_\sigma(\gamma_c),L_3,L_4) \qquad (73) \\
R(\sigma) &= \sum_{c\in\mathfrak{C}_{b_1,b_2}(\mathbf{\Sigma})} \mathsf{B}(L_1,L_2,\ell_\sigma(\gamma_c))\mathsf{B}(\ell_\sigma(\gamma_c),L_3,L_4) + \sum_{c\in\mathfrak{C}_{b_1,b_3}(\mathbf{\Sigma})} \mathsf{B}(L_1,L_3,\ell_\sigma(\gamma_c))\mathsf{B}(L_2,\ell_\sigma(\gamma_c),L_4) \\
&\quad + \sum_{c\in\mathfrak{C}^{b_4}_{b_1,b_1}(\mathbf{\Sigma})} \mathsf{C}(L_1,L_4,\ell_\sigma(\gamma_c)).
\end{aligned}
$$

Is it true that Q *and* R *are invariant under braiding of* b_1 *and* b_2 *?*

If the answer to this question is positive, it would explain by the general mechanism of Theorem 5.3 why a priori Ω_Σ for Mirzakhani initial data is invariant under braiding of all boundaries – without knowing that it is in fact equal to 1 for all Σ.

We know that the strict Mirzakhani initial data are symmetric, because they are equivalently described in Laplace transform and we can then rely on Lemma 7.5. In particular, we know that

Lemma 10.9 *If* Σ *is a sphere with four boundaries, for any* $\mathsf{X} \in \{\mathsf{Q},\mathsf{R}\}$ *we have*

$$\int_{\mathcal{M}_\Sigma(L_1,L_2,L_3,L_4)} \mathsf{X}(\sigma) = \int_{\mathcal{M}_\Sigma(L_2,L_1,L_3,L_4)} \mathsf{X}(\sigma).$$

Equivalently,

$$
\begin{aligned}
\int_{\mathbb{R}_{>0}} \mathrm{d}\ell\,\ell\big(\mathsf{B}(L_1,L_2,\ell)\mathsf{C}(\ell,L_3,L_4) + \mathsf{C}(L_1,L_3,\ell)\mathsf{B}(L_2,\ell,L_4) + \mathsf{C}(L_1,L_4,\ell)B(L_2,\ell,L_3)\big) &= (L_1 \leftrightarrow L_2), \\
\int_{\mathbb{R}_{>0}} \mathrm{d}\ell\,\ell\big(\mathsf{B}(L_1,L_2,\ell)\mathsf{B}(\ell,L_3,L_4) + \mathsf{B}(L_1,L_3,\ell)\mathsf{B}(L_2,\ell,L_4) + \mathsf{C}(L_1,L_4,\ell)\big) &= (L_1 \leftrightarrow L_2).
\end{aligned}
$$

■

A positive answer to the above question is tantamount to a finer version of Lemma 10.9. Lemma 10.9 would then be a corollary obtained by averaging over the length ℓ of the closed curves along which one can cut a sphere with four boundaries. To answer positively the question, it would for instance be enough to prove that $(\mathrm{Id} - \sigma_{b_1,b_2})$ applied to (73) is a constant function over the Teichmüller space.

10.5 The relation $B + B = C + A$

A direct computation shows that Mirzakhani initial data satisfies

$$\underline{\mathsf{B}}(L_1,L_2,L_3) + \underline{\mathsf{B}}(L_1,L_3,L_2) = \underline{\mathsf{C}}(L_1,L_2,L_3) + \underline{\mathsf{A}}(L_1,L_2,L_3) \qquad (74)$$

with here $\underline{\mathsf{A}}(L_1, L_2, L_3) = 1$. This relation is explained by the geometric origin of these functions in the work of Mirzakhani [40]. For any object $\mathbf{\Sigma}$ in $\mathrm{Bord}_1^{\bullet}$, an important ingredient in the proof of Theorem 10.1 is a decomposition of $\partial_-\Sigma$ into subsets

$$\partial_-\Sigma = \mathscr{B}_{\Sigma}(\sigma) \cup \mathscr{C}_{\Sigma}(\sigma) \cup \mathscr{Z}_{\Sigma}(\sigma) \tag{75}$$

which are measurable for the Lebesgue measure (denoted μ) with respect to the arc-length parametrisation of $\partial_-\Sigma$ induced by σ. By definition, an orthogeodesic is a geodesic in Σ which is orthogonal to $\partial\Sigma$ at each of its point of intersection with $\partial\Sigma$. Let γ_x be the geodesic shot from $x \in b_1$ orthogonally to b_1.

- $\mathscr{B}_{\Sigma}(\sigma)$ is the set of $x \in \partial_-\Sigma$ such that the homotopy (relative to the boundary) class c_x of γ_x belongs to $\mathfrak{C}_{b_1,b_i}(\mathbf{\Sigma})$ for some $b_i \in \pi_0(\partial_+\Sigma)$.
- $\mathscr{C}_{\Sigma}(\sigma)$ is the set of $x \in \partial_-\Sigma$ such that the homotopy class of γ_x belongs to $\mathfrak{C}_{b_1,b_1}(\mathbf{\Sigma})$
- the complement set $\mathscr{Z}_{\Sigma}(\sigma)$ has measure zero.

The sets $\mathscr{B}_{\Sigma}(\sigma)$ and $\mathscr{C}_{\Sigma}(\sigma)$ can be further partitioned by their intersection with the fibers of $\mathsf{c} : \partial_-\Sigma \to \mathfrak{C}(\mathbf{\Sigma})$ which associates c_x to x, *i.e.* equivalently the homotopy class of embedded pair of pants determined by γ_x.

The measure of each of this piece can be computed purely in terms of the geometry of a hyperbolic pair of pants P embedded in Σ with exactly one negatively oriented boundary component ∂_-P. There are seven special orthogeodesics on P (see Figure 19): one from $p_1 \in b_1$ to $p_2 \in b_1$, one from $r \in b_1$ to b, one from $r' \in b_1$ to b', one starting from $o_1 \in b_1$ and spiralling around b, one from $o_1' \in b_1$ and spiralling around b', and the ones related to the last two by exchanging the role of the two hexagons which glue to form P. For any $c \in \mathfrak{C}(\mathbf{\Sigma})$, one has

$$\begin{aligned} \mu\big[\mathscr{B}_{\Sigma}(\sigma) \cap \mathsf{c}^{-1}(c)\big] &= \mu\big[\overline{I}(P_c(\sigma))\big] = \underline{\mathsf{B}}(\ell_\sigma(b_1), \ell_\sigma(b), \ell_\sigma(b')) \\ \mu\big[\mathscr{C}_{\Sigma}(\sigma) \cap \mathsf{c}^{-1}(c)\big] &= \mu\big[I_1(P_c(\sigma)) \cup I_2(P_c(\sigma))\big] = \underline{\mathsf{C}}(\ell_\sigma(b_1), \ell_\sigma(b), \ell_\sigma(b')) \end{aligned}$$

with the notation of Figure 19, and hyperbolic trigonometry led Mirzakhani to the expressions (59) for $\underline{\mathsf{B}}$ and $\underline{\mathsf{C}}$. And from the decomposition of $\partial_-P = b_1$ in the figure, we see that these functions must satisfy (74) and

$$\underline{\mathsf{A}}(L_1, L_2, L_3) = 1$$

appears as the total mass of μ.

We already met a similar relation (74) on initial data, in the context of the topological recursion –Section 7.4 and Lemma 7.8. It although appeared in different variables z_i instead of L_i – which are in fact Laplace dual (see Section 11).

Question 10.10 *Can one give a similar, geometric interpretation of the relation $B + B = C + A$ of Lemma 7.8, which we know holds for the initial data attached to any spectral curve ?*

11 Laplace transforms of the boundary lengths

There are many equivalent ways to describe continuous functions on $\mathbb{R}^n_{>0}$. One of them is via their Laplace transform. In this section, we explore these alternative representations for admissible initial data. The slogan is that the Laplace variable which is dual to boundary length should be identified,

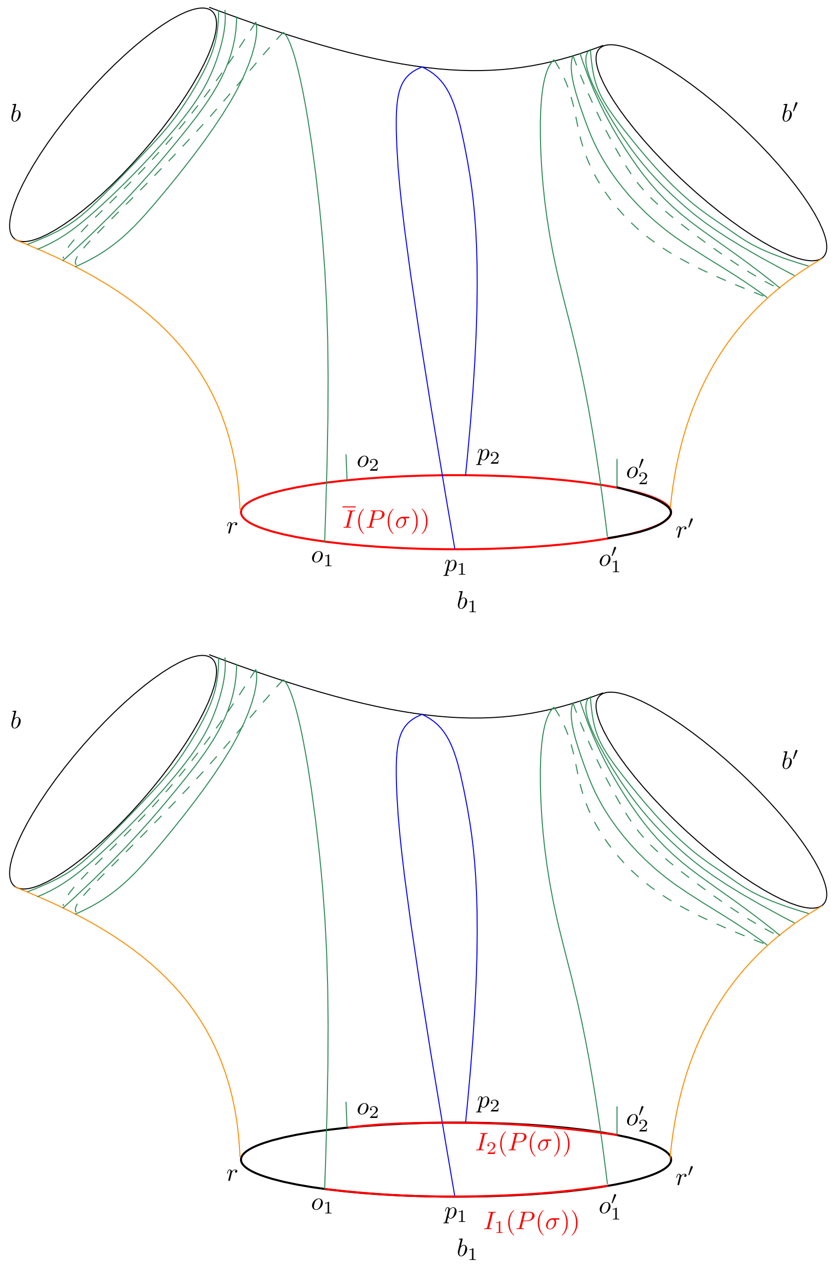

Figure 19: Identification of the subset $\mathscr{X}_\Sigma(\sigma) \cap \mathsf{c}^{-1}(c)$ in P_c, in the case $\mathscr{X} = \mathscr{B}$ (top) or $\mathscr{X} = \mathscr{C}$ (bottom).

in terms of the original topological recursion on spectral curves, with the local variable z near a ramification point, in which the deck transformation acts as $z \mapsto -z$. However, convergence subtleties prevents from using the results of this section to describe the original topological recursion in terms of length variables. For this we will have to wait for Section 12.

At an abstract level, the Laplace transform is better defined in terms of distributions. As we prefer to avoid unnecessary technicalities in this computationally-minded section, we will keep ourselves at the level of functions with a few (maybe non optimal) regularity assumptions. We refer to the monograph [53] for the properties of the Laplace transform which we use.

11.1 The glueing morphism in Laplace variable

If U is an open subset of $\mathbb{C}$, we denote $\mathscr{O}(U)$ the space of holomorphic functions on U. If $t \in \mathbb{R}$, we introduce

$$U_t := \{z \in \mathbb{C} \quad : \quad \mathrm{Re}\, z > t\}.$$

The Laplace transform of a function $f : \mathbb{R}_{>0} \to \mathbb{C}$ is defined by

$$\widehat{f}(z) := \mathcal{L}[f](z) := \int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, e^{-\ell z} f(\ell)$$

when it exists. The definition extends to functions of n positive real variables.

Let $t \in \mathbb{R}$ and set $\mathrm{e}_t(\ell) := e^{t\ell}$. Recall from Definition 9.1 that the space $\mathcal{C}^0_{\mathrm{e}_t\cdot\mathrm{poly}}(\mathbb{R}_{>0})$ consists of continuous functions which are bounded at 0, and are bounded at infinity by $\ell^r e^{t\ell}$ for some $r > 0$. The Laplace transform gives an injective linear map

$$\mathcal{L} : \mathcal{C}^0_{\mathrm{e}_t\cdot\mathrm{poly}}(\mathbb{R}^n) \longrightarrow \mathscr{O}(U_t^n).$$

We rely on the basic Laplace inversion result.

Lemma 11.1 *If $f \in \mathcal{C}^0_{\mathrm{e}_t\cdot\mathrm{poly}}(\mathbb{R}_{>0})$, for any $t'' > t$ we have,*

$$\forall \ell > 0, \qquad f(\ell) = \frac{1}{2\mathrm{i}\pi} \int_{t''+\mathrm{i}\mathbb{R}} \mathrm{d}z\, e^{z\ell}\, \widehat{f}(z)$$

where the right-hand side is defined as an improper integral, i.e. the limit of $\int_{t''-\mathrm{i}M}^{t''+\mathrm{i}M'}$ when $M, M' \to \infty$. For any $\ell < 0$, the integral in the right-hand side evaluates to 0.

Corollary 11.2 *Let $t' > t$. If $f \in \mathcal{C}^0_{\mathrm{e}_{-t'}\cdot\mathrm{poly}}(\mathbb{R}_{>0})$ and $g \in \mathcal{C}^0_{\mathrm{e}_t\cdot\mathrm{poly}}(\mathbb{R}_{>0})$, then $\ell \mapsto \ell\, f(\ell) g(\ell)$ is integrable on $\mathbb{R}_{>0}$. Let $t'' \in (t, t')$ and assume that g is $\mathcal{C}^1$ and $(g \cdot \mathrm{e}_{t''})'$ has bounded variation on $\mathbb{R}_{>0}$,*

$$\int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, f(\ell) g(\ell) = -\frac{1}{2\mathrm{i}\pi} \int_{t''+\mathrm{i}\mathbb{R}} \mathrm{d}z\, \widehat{f}(-z)\, \widehat{g}'(z) = -\frac{1}{2\mathrm{i}\pi} \int_{t''+\mathrm{i}\mathbb{R}} \mathrm{d}z\, \widehat{f}'(-z)\, \widehat{g}(z).$$

Proof. The growth assumptions imply that $f \cdot g$ is integrable. We apply Lemma 11.1 and write for $t'' > t$ and $\ell > 0$

$$\int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, f(\ell)\, g(\ell) = -\frac{1}{2\mathrm{i}\pi} \int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, f(\ell) \int_{t''+\mathrm{i}\mathbb{R}} \mathrm{d}z\, e^{z\ell}\, \widehat{g}'(z)$$

We have

$$\left|f(\ell) e^{z\ell} \widehat{g}'(z)\right| \le |f(\ell)| e^{(\mathrm{Re}\, z)\ell} |\widehat{g}'(z)|. \tag{76}$$

To estimate the growth when $\mathrm{Im}\, z \to \infty$ while $\mathrm{Re}\, z = t''$, we rely on the basic result

Lemma 11.3 *If $f \in \mathcal{C}^0_{\mathrm{e}_t\cdot\mathrm{poly}}(\mathbb{R})$ has bounded variation, for any $t'' > t$ we have*

$$\sup_{z \in t''+\mathrm{i}\mathbb{R}} \left|z\, \widehat{f}(z)\right| < +\infty.$$

□

Consequently, the assumptions on g imply that $\sup_{z \in t''+\mathrm{i}\mathbb{R}} |z^2 \widehat{g}'(z)| < +\infty$. We deduce that (76) is integrable over $\mathbb{R}_{>0} \times (t'' + \mathrm{i}\mathbb{R})$ provided we choose $t'' < t'$. By Fubini theorem, we then find

$$\int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, f(\ell)\, g(\ell) = -\frac{1}{2\mathrm{i}\pi} \int_{t''+\mathrm{i}\mathbb{R}} \mathrm{d}z \left(\int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, e^{z\ell}\, f(\ell) \right) \widehat{g}'(z) = -\frac{1}{2\mathrm{i}\pi} \int_{t''+\mathrm{i}\mathbb{R}} \mathrm{d}z\, \widehat{f}(-z) \widehat{g}'(z).$$

■

11.2 Polynomials in ℓ and rational functions in z

In this paragraph, we want to describe a situation where the integral over z in Corollary 11.2 can be turned into a sum of residues, so as to bring the setup of strict GR closer to the original setup of the topological recursion in terms of local spectral curves. The latter will be compared in the next paragraph.

We introduce two vector spaces depending on the data of a cartesian product of n subsets of $\mathbb{C}$, here denoted $\mathbf{\Lambda}$. The first space is

$$\mathscr{F}_{\mathbf{\Lambda}} := \bigoplus_{\lambda \in \mathbf{\Lambda}} \left(\mathbb{C}[L_1, \ldots, L_n] \cdot \prod_{i=1}^{n} e^{\lambda_i L_i} \right).$$

Elements in the direct sum are finite linear combinations of elements in the summands. We will avoid discussing its topological completion as it would blur our illustrative purposes. The Laplace transforms induces an isomorphism from $\mathscr{F}_{\mathbf{\Lambda}}$ to a second space, which we denote $\mathscr{M}_{\mathbf{\Lambda}}$. It consists of the rational functions $\varphi(z_1, \ldots, z_n)$ of n variables, which can have poles when z_i approaches $\mathrm{pr}_i(\mathbf{\Lambda})$, and are such that

$$\varphi(z_1, \ldots, z_n) \in O(z_1^{-1} \cdots z_n^{-1}), \qquad z_i \to \infty.$$

In the following, we fix two subsets $\Lambda, \Lambda_B \subset \mathbb{C}$, and denote

$$t = \sup_{\lambda \in \Lambda} \mathrm{Re}\,\lambda, \qquad t' := \inf_{\beta \in \Lambda_B} \mathrm{Re}\,\beta$$

and assume

$$t < t'. \tag{77}$$

By convention $-\Lambda_B := \{-\beta \; : \; \beta \in \Lambda_B\}$. Assume we are given functions

$$B \in \mathscr{F}_{\Lambda^2 \times (-\Lambda_B)}, \qquad C \in \mathscr{F}_{\Lambda \times (-\Lambda_B)^2}. \tag{78}$$

For any $f \in \mathscr{F}_{\Lambda}$, the assumption (77) implies that $B(L_1, L_2, \ell) f(\ell)$ has exponential decay when $\ell \to \infty$. Therefore, B and f are glueable, and we obtain an expression for the glueing solely in terms of Laplace variables (see Figure 20).

Lemma 11.4 *For any $f \in \mathscr{F}_{\Lambda}$, B and f are glueable, and*

$$\begin{aligned} \int_{\mathbb{R}_{>0}^2} \mathrm{d}L_1\, \mathrm{d}L_2\, e^{-z_1 L_1 - z_2 L_2} \int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, B(L_1, L_2, \ell)\, f(\ell) &= \operatorname*{Res}_{z \to \Lambda_B} \widehat{B}(z_1, z_2, -z)\, \mathrm{d}_z \widehat{f}(z) \\ &= -\operatorname*{Res}_{z \to \Lambda} \widehat{B}(z_1, z_2, -z)\, \mathrm{d}_z \widehat{f}(z)\,. \end{aligned} \tag{79}$$

For any $f \in \mathscr{F}_{\Lambda^2}$, C and f are glueable, and

$$\begin{aligned} \int_{\mathbb{R}_{>0}} \mathrm{d}L_1\, e^{-z_1 L_1} \int_{\mathbb{R}_{>0}^2} \mathrm{d}\ell\, \ell\, \mathrm{d}\tilde{\ell}\, \tilde{\ell}\, C(L_1, \ell, \tilde{\ell})\, f(\ell, \tilde{\ell}) &= \operatorname*{Res}_{\substack{z \to \Lambda_B \\ \tilde{z} \to \Lambda_B}} \widehat{C}(z_1, -z, -\tilde{z})\, \mathrm{d}_z \mathrm{d}_{\tilde{z}} \widehat{f}(z, \tilde{z}) \\ &= \operatorname*{Res}_{\substack{z \to \Lambda \\ \tilde{z} \to \Lambda}} \widehat{C}(z_1, -z, -\tilde{z})\, \mathrm{d}_z \mathrm{d}_{\tilde{z}} \widehat{f}(z, \tilde{z})\,. \end{aligned} \tag{80}$$

Proof. The assumptions of Corollary 11.2 are satisfied, and we have, for $z_1, z_2 \in U_t$ and $t'' \in (t, t')$,

$$\int_{\mathbb{R}_{>0}} \mathrm{d}L_1 \mathrm{d}L_2\, e^{-z_1 L_1 - z_2 L_2} \left(\int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, B(L_1, L_2, \ell) f(\ell) \right) = -\frac{1}{2\mathrm{i}\pi} \int_{t'' + \mathrm{i}\mathbb{R}} \mathrm{d}z\, \widehat{B}(-z, z_1, z_2) \mathrm{d}_z \widehat{f}(z).$$

>From the definition of our spaces of functions, we know that $\widehat{f}'(z) \in O(z^{-2})$ when $z \to \infty$, and $\widehat{B}(z_1, z_2, -z)$ is $O(z^{-1})$ when $z \to \infty$. Therefore, we can close the contour and surround all the poles

in the region $U_{t''}$. The assumption (77) implies that $\widehat{f}(z)$ has no poles in this region, so we are picking up the residues at the poles of $\widehat{B}(z_1, z_2, z)$ in the variable z, which occur at $-\Lambda_B$. The formula (80) is derived similarly. The second line in each formula is obtained by closing the contour to surround all the complement of $U_{t''}$. ∎

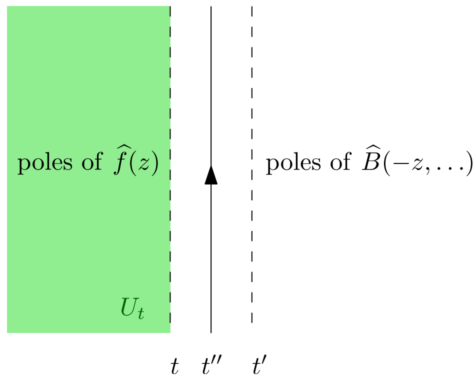

Figure 20: Poles in the Laplace variable.

12 Lifting TR initial data to GR

12.1 Interpolating between Mirzakhani and Witten-Kontsevich

We start by describing a deformation of Mirzakhani initial data, which consists in rescaling all lengths by a factor $\beta > 0$. Let us define

$$\mathsf{F}_\beta(x) = 2\ln\left(1 + e^{-\frac{\beta x}{2}}\right), \qquad \mathsf{F}(x) := \mathsf{F}_1(x). \tag{81}$$

In the language of statistical physics, the parameter $\beta > 0$ is interpreted as an inverse temperature, and F_β is closely related to the Fermi-Dirac weight. We now introduce

$$\begin{aligned}
\mathsf{A}_\beta(L_1, L_2, L_3) &:= 1, \\
\mathsf{B}_\beta(L_1, L_2, \ell) &:= \frac{1}{2\beta L_1}\big(\mathsf{F}_\beta(\ell + L_2 - L_1) + \mathsf{F}_\beta(\ell - L_2 - L_1) - \mathsf{F}_\beta(\ell - L_2 + L_1) - \mathsf{F}_\beta(\ell + L_2 + L_1)\big), \\
\mathsf{C}_\beta(L_1, \ell, \ell') &:= \frac{1}{\beta L_1}\big(\mathsf{F}_\beta(\ell + \ell' - L_1) - \mathsf{F}_\beta(\ell + \ell' + L_1)\big), \\
\mathsf{D}_\beta(L_1) &:= \sum_{\gamma \in \hat{S}^\circ_{\Sigma_{1,1}}} \mathsf{C}_\beta(L_1, \ell_\sigma, \ell_\sigma).
\end{aligned}$$

The choice for D_β is the natural deformation of (63), and is justified in order to have Lemma 12.1 below. Since $\mathsf{F}_\beta(x)/\beta$ is a decreasing function of $\beta > 0$ for any $x \in \mathbb{R}$, we deduce that $(\mathsf{A}_\beta, \mathsf{B}_\beta, \mathsf{C}_\beta, \mathsf{D}_\beta)$ is uniformly M-bounded (see Definition 10.6) for any $\beta \in [1, +\infty]$.

In fact,

$$\mathsf{G}(x) := \lim_{\beta \to \infty} \frac{\mathsf{F}_\beta(-x)}{\beta} = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{82}$$

and the convergence is uniform for $x \in \mathbb{R}$. As lengths are positive, we have

$$\begin{aligned}
\mathsf{B}_\infty(L_1, L_2, \ell) &:= \lim_{\beta \to \infty} \mathsf{B}_\beta(L_1, L_2, \ell) = \frac{1}{2L_1}\big(\mathsf{G}(L_1 - L_2 - \ell) - \mathsf{G}(-L_1 + L_2 - \ell) + \mathsf{G}(L_1 + L_2 - \ell)\big), \\
\mathsf{C}_\infty(L_1, \ell, \ell') &:= \lim_{\beta \to \infty} \mathsf{C}_\beta(L_1, \ell, \ell') = \frac{1}{L_1}\mathsf{G}(L_1 - \ell - \ell')
\end{aligned}$$

and the convergence is uniform over $\mathbb{R}^3_{>0}$. One can check directly that these functions are nonnegative. These remarks on uniformity allow to conclude that computing the GR amplitudes for the $\beta \to \infty$ initial data gives the same result as taking the limit $\beta \to \infty$ of the GR amplitudes.

Let $\Omega_{\mathbf{\Sigma},\beta}$ be the GR amplitudes for the initial data $(\mathsf{A}_\beta, \mathsf{B}_\beta, \mathsf{C}_\beta, \mathsf{D}_\beta)$. We do not have *a priori* a closed formula for $\Omega_{\mathbf{\Sigma},\beta}$, as rescaling all lengths in (62) – including the lengths of the curve on which we glue when excising a pair of pants – does not have an obvious geometric interpretation. However, we can describe the outcome of this rescaling after integration over the moduli space. Let us introduce

$$\mathsf{V}_{g,n,\beta}(L) = \int_{\mathcal{M}_{\mathbf{\Sigma}_{g,n}}(L)} \Omega_{\mathbf{\Sigma}_{g,n},\beta} \cdot \nu_{\Sigma_{g,n}}$$

and the following initial data for the strict GR

$$\begin{aligned} \underline{\mathsf{A}}_\beta(L_1, L_2, L_3) &= 1, \\ \underline{\mathsf{B}}_\beta(L_1, L_2, \ell) &= \mathsf{B}_\beta(L_1, L_2, \ell), \\ \underline{\mathsf{C}}_\beta(L_1, \ell, \ell') &= \mathsf{C}_\beta(L_1, \ell, \ell'), \\ \underline{\mathsf{D}}_\beta(L_1) &= \frac{\pi^2}{6\beta^2} + \frac{L_1^2}{24}. \end{aligned} \tag{83}$$

Lemma 12.1 *For any $\beta > 0$ and $2g - 2 + n > 0$, we have*

$$\mathsf{V}_{g,n,\beta}(L_1, \dots, L_n) = \beta^{-(6g-6+2n)}\, \mathsf{V}_{g,n}(\beta L_1, \dots, \beta L_n) \tag{84}$$

which is also equal to the strict GR amplitude for the initial data $(\underline{\mathsf{A}}_\beta, \underline{\mathsf{B}}_\beta, \underline{\mathsf{C}}_\beta, \underline{\mathsf{D}}_\beta)$.

Proof. The $(g,n) = (0,3)$ case is obvious. We first consider the case $(g,n) = (1,1)$. We integrate (83) and find

$$\int_{\mathcal{M}_{\Sigma_{1,1}}(L)} \mathsf{D}_\beta \cdot \nu_{\Sigma_{1,1}} = \int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, \mathsf{C}_\beta(L_1, \ell, \ell) = \beta^{-2} \int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, \mathsf{C}(\beta L_1, \ell, \ell) = \beta^{-2}\Big(\tfrac{\pi^2}{6} + \tfrac{\beta^2 L_1^2}{24}\Big)$$

where we have used the change of variable $\ell \mapsto \beta\ell$, and the result of integration is given (or directly recomputed) by $\beta^{-2}\underline{\mathsf{D}}(\beta L_1) = \underline{\mathsf{D}}_\beta(L_1)$. This justifies that $V_{1,1,\beta}$ is equal to $\underline{\mathsf{D}}_\beta(L_1)$, which is also equal by definition to the strict GR amplitude for $(g,n) = (1,1)$.

We then prove the result by induction on $2g - 2 + n \geq 2$. Assume it holds for any g', n' such that $2g' - 2 + n' < 2g - 2 + n$. The strict GR amplitude for (g,n) is equal to

$$\begin{aligned} &\sum_{m=2}^{n} \int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, \mathsf{B}_\beta(L_1, L_2, \ell)\, \mathsf{V}_{g,n-1,\beta}(\ell, L_2, \dots, \widehat{L_m}, \dots, L_n) \\ &+ \frac{1}{2} \int_{\mathbb{R}^2_{>0}} \mathrm{d}\ell\, \mathrm{d}\ell'\, \ell\, \ell' \mathsf{C}_\beta(L_1, \ell, \ell') \bigg(\mathsf{V}_{g-1,n+1,\beta}(\ell, \ell', L_2, \dots, L_n) + \sum_{\substack{g_1+g_2=g \\ J_1 \cup J_2 = \{L_2, \dots, L_n\}}} \mathsf{V}_{g_1,1+|J_1|,\beta}(\ell, J_1) \mathsf{V}_{g_2,1+|J_2|,\beta}(\ell', J_2) \bigg) \end{aligned}$$

We use the induction hypothesis by substituting (84) for all V's in this expression, and perform the change of variable of integration $(\ell, \ell') \to (\beta\ell, \beta\ell')$. As we have

$$\mathsf{B}_\beta(L_1, L_2, \ell) = \mathsf{B}(\beta L_1, \beta L_2, \beta\ell), \qquad \mathsf{C}_\beta(L_1, \ell, \ell') = \mathsf{C}(\beta L_1, \beta\ell, \beta\ell'),$$

we recognise the formula for the strict GR amplitude for the initial data $(\underline{\mathsf{A}}_1, \underline{\mathsf{B}}_1, \underline{\mathsf{C}}_1, \underline{\mathsf{D}}_1)$, multiplied by an overall factor $\beta^{-(6g-6+2n)}$. As explained in Section 10.1 it is also the expression for $\beta^{-(6g-6+2n)}\mathsf{V}_{g,n}(L)$. This concludes the proof.

Note that $3g - 3 + n$ is the number of interior curves which bound pair of pants in a decomposition of $\Sigma_{g,n}$. The factor of $\beta^{-(6g-6+2n)}$ comes from the fact that we have to integrate over length of all the curves the measure $\ell\, \mathrm{d}\ell$. ∎

Corollary 12.2 *For $2g-2+n>0$, we have equality of the three following quantities.*

(1) *the integration, against the Weil-Petersson volume form ν_Σ,*

$$\mathsf{V}_{g,n,\infty}(L_1,\ldots,L_n) = \int_{\mathcal{M}_{\Sigma_{g,n}}(L)} \Omega_{\Sigma_{g,n},\infty}\cdot \nu_{\Sigma_{g,n}}$$

over the moduli space of bordered surfaces of the GR amplitudes for the initial data

$$\begin{aligned}
\mathsf{A}_{\mathbf{P},\infty}(\sigma) &= 1 \\
\mathsf{B}^b_{\mathbf{P},\infty}(\sigma) &= \frac{1}{2\ell_\sigma(b_1)}\Big\{\mathsf{G}(\ell_\sigma(b_1)-\ell_\sigma(b)-\ell_\sigma(b')) - \mathsf{G}(-\ell_\sigma(b_1)+\ell_\sigma(b)-\ell_\sigma(b')) + \mathsf{G}(\ell_\sigma(b_1)+\ell_\sigma(b)-\ell_\sigma(b'))\Big\} \\
\mathsf{C}_{\mathbf{P},\infty}(\sigma) &= \frac{1}{\ell_\sigma(b_1)}\,\mathsf{G}(\ell_\sigma(b_1)-\ell_\sigma(b)-\ell_\sigma(b'))
\end{aligned}$$

Here $\mathbf{P}$ *is a pair of pants with boundary components $\partial_- P = b_1$ and $\partial_+ P = \{b,b'\}$, and D is induced by Lemma 4.3.*

(2) *the strict GR amplitudes for the initial data*

$$\begin{aligned}
\underline{\mathsf{A}}_\infty(L_1,L_2,L_3) &= 1 \\
\underline{\mathsf{B}}_\infty(L_1,L_2,\ell) &= \frac{1}{2L_1}\big(\mathsf{G}(L_1-L_2-\ell) - \mathsf{G}(-L_1+L_2-\ell) + \mathsf{G}(L_1+L_2-\ell)\big) \\
\underline{\mathsf{C}}_\infty(L_1,\ell,\ell') &= \frac{1}{L_1}\,\mathsf{G}(L_1-\ell-\ell') \\
\underline{\mathsf{D}}_\infty(L_1) &= \frac{L_1^2}{24}
\end{aligned}$$

(3) *the generating series of ψ-classes intersections on the moduli space of curves,*

$$\sum_{\substack{d_1,\ldots,d_n\geq 0 \\ d_1+\cdots+d_n = 3g-3+n}} \bigg(\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{d_i}\bigg) \prod_{i=1}^n \frac{L_i^{2d_i}}{2^{d_i} d_i!}.$$

Proof. The equality of (1) and (2) comes from the variant of Proposition 9.4 for M-bounded stated in Proposition 10.7, and we already stressed that taking the $\beta\to\infty$ limit in the initial data of GR due to the uniformity of estimates. The last equality comes from the observation that taking the limit $\beta\to\infty$ in $\mathsf{V}_{g,n,\beta}(L)$ amounts to selecting the top degree ($=3g-3+n$) term from the ψ-classes in Theorem 10.2. ∎

12.2 Witten-Kontsevich model in Laplace variable

Section 11 does not apply to initial data which are M-admissible but maybe not admissible, because in such a case B and C fails to satisfy (78). We now explain how to treat the fundamental example of the Witten-Kontsevich model in Laplace variables, starting from Corollary 12.2. This will allow us in the next paragraphs to lift any initial data of TR (in Laplace variable) to an initial data of GR (in length variables), such that integration of the GR amplitudes over the moduli spaces retrieves, after Laplace transform, the TR amplitudes.

Recall from Section 12.1 the definitions

$$\underline{\mathsf{B}}_\infty(L_1,L_2,\ell) = \frac{1}{2L_1}\big(\mathsf{G}(L_1-L_2-\ell) - \mathsf{G}(-L_1+L_2-\ell) + \mathsf{G}(L_1+L_2-\ell)\big), \tag{85}$$

$$\underline{\mathsf{C}}_\infty(L_1,\ell,\ell') = \frac{1}{L_1}\,\mathsf{G}(L_1-\ell-\ell'), \tag{86}$$

where $\mathsf{G}(x)=\max(x,0)$. We first evaluate separately the Laplace transform of each term.

Lemma 12.3 *Let $f_m \in \mathcal{C}^0_{\mathrm{poly}}(\mathbb{R}^m_{>0})$ for $m \in \{1,2\}$. For $\mathrm{Re}\, z_1 > 0$ and $\mathrm{Re}\, z_2 > 0$, we have*

$$\begin{aligned}
\int_{\mathbb{R}^2_{>0}} \mathrm{d}L_1\, \mathrm{d}L_2\, L_1\, L_2\, e^{-z_1 L_1 - z_2 L_2} \int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, \frac{\mathsf{G}(L_1 + L_2 - \ell)}{L_1}\, f_1(\ell) &= -\partial_{z_2}\bigg(\frac{z_1^{-2}\widehat{f}_1'(z_1) - z_2^{-2}\widehat{f}_1'(z_2)}{z_1 - z_2}\bigg), \\
\int_{\mathbb{R}^2_{>0}} \mathrm{d}L_1\, \mathrm{d}L_2\, L_1\, L_2\, e^{-z_1 L_1 - z_2 L_2} \int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, \frac{\mathsf{G}(L_1 - L_2 - \ell)}{L_1}\, f_1(\ell) &= \partial_{z_2}\bigg(\frac{\widehat{f}_1'(z_1)}{z_1^2(z_1 + z_2)}\bigg), \\
\int_{\mathbb{R}^2_{>0}} \mathrm{d}L_1\, \mathrm{d}L_2\, L_1\, L_2\, e^{-z_1 L_1 - z_2 L_2} \int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, \frac{\mathsf{G}(-L_1 + L_2 - \ell)}{L_1}\, f_1(\ell) &= \partial_{z_2}\bigg(\frac{\widehat{f}_1'(z_2)}{z_2^2(z_1 + z_2)}\bigg), \\
\int_{\mathbb{R}_{>0}} \mathrm{d}L_1\, L_1\, e^{-z_1 L_1} \int_{\mathbb{R}^2_{>0}} \mathrm{d}\ell\, \mathrm{d}\ell'\, \ell\, \ell'\, \frac{\mathsf{G}(L_1 - \ell - \ell')}{L_1}\, f_2(\ell, \ell') &= z_1^{-2}\partial_1\partial_2\widehat{f}_2(z_1, z_1).
\end{aligned}$$

Proof. The last equality is easy, since the left-hand side is equal to

$$\begin{aligned}
\int_{\mathbb{R}^2_{>0}} \mathrm{d}\ell\, \mathrm{d}\ell'\, \ell\, \ell'\, f_2(\ell, \ell') \int_{\ell + \ell'}^{\infty} \mathrm{d}L_1\, e^{-z_1 L_1}\, (L_1 - \ell - \ell') &= \int_{\mathbb{R}^2_{>0}} \mathrm{d}\ell \mathrm{d}\ell'\, \ell\, \ell'\, f_2(\ell, \ell')\, z_1^{-2}\, e^{-z_1(\ell + \ell')} \\
&= z_1^{-2}\, \partial_1\partial_2\widehat{f}_2(z_1, z_1)
\end{aligned}$$

The first three equalities require more computations. We introduce:

$$\begin{aligned}
\mathsf{J}_+(z_1, z_2; \ell) &= \int_{\mathbb{R}^2_{>0}} \mathrm{d}L_1 \mathrm{d}L_2\, e^{-z_1 L_1 - z_2 L_2}\, \mathsf{G}(L_1 + L_2 - \ell), \\
\mathsf{J}_-(z_1, z_2, \ell) &= \int_{\mathbb{R}^2_{>0}} \mathrm{d}L_1\, \mathrm{d}L_2\, e^{-z_1 L_1 - z_2 L_2}\, \mathsf{G}(L_1 - L_2 - \ell).
\end{aligned}$$

The left-hand sides of the three first lines of the claim are respectively equal to

$$-\partial_{z_2}\bigg(\int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, \mathsf{J}_+(z_1, z_2, \ell)\, f_1(\ell)\bigg), \quad -\partial_{z_2}\bigg(\int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, \mathsf{J}_-(z_1, z_2, \ell)\, f_1(\ell)\bigg) \quad \text{and} \quad -\partial_{z_2}\bigg(\int_{\mathbb{R}_{>0}} \mathrm{d}\ell\, \ell\, \mathsf{J}_-(z_2, z_1, \ell)\, f_1(\ell)\bigg).$$

We compute

$$\begin{aligned}
\mathsf{J}_+(z_1, z_2; \ell) &= \int_0^{\ell} \mathrm{d}L_1\, e^{-z_1 L_1} \int_{\ell - L_1}^{\infty} \mathrm{d}L_2\, e^{-z_2 L_2}\, (L_1 + L_2 - \ell) + \int_{\ell}^{\infty} \mathrm{d}L_1\, e^{-z_1 L_1} \int_0^{\infty} \mathrm{d}L_2\, e^{-z_2 L_2} (L_1 + L_2 - \ell) \\
&= \int_0^{\ell} \mathrm{d}L_1\, e^{-z_1 L_1}\, \frac{e^{-z_2(\ell - L_1)}}{z_2^2} + \int_{\ell}^{\infty} \mathrm{d}L_1\, e^{-z_1 L_1}\bigg(\frac{L_1 - \ell}{z_2} + \frac{1}{z_2^2}\bigg) \\
&= \frac{e^{-z_1\ell} - e^{-z_2\ell}}{(z_1 - z_2) z_2^2} + \frac{z_1 + z_2}{z_1^2 z_2^2}\, e^{-z_1 L} \\
&= -\frac{1}{z_1 - z_2}\bigg(\frac{e^{-z_1\ell}}{z_1^2} - \frac{e^{-z_2\ell}}{z_2^2}\bigg),
\end{aligned}$$

and

$$\mathsf{J}_-(z_1, z_2; \ell) = \int_0^{\infty} \mathrm{d}L_2\, e^{-z_2 L_2} \int_{L_2 + \ell}^{\infty} \mathrm{d}L_1\, e^{-z_1 L_1}\, (L_1 - L_2 - \ell) = \int_0^{\infty} \mathrm{d}L_2\, e^{-z_2 L_2}\, \frac{e^{-z_1(L_2 + \ell)}}{z_1^2} = \frac{e^{-z_1\ell}}{z_1^2(z_1 + z_2)}$$

from which the identities follow. ∎

We can then supplement the equivalences stated in Corollary 12.2.

Proposition 12.4 *We have equality of the three following quantities, for $2g - 2 + n > 0$*

(1) *the Laplace transforms, defined for $\mathrm{Re}\, z_i > 0$ by the formula*

$$\int_{\mathbb{R}^n_{>0}} \prod_{i=1}^{n} \mathrm{d}L_i\, L_i\, e^{-z_i L_i}\, \mathsf{V}_{g,n,\infty}(L_1, \ldots, L_n); \tag{87}$$

(2) *the generating series of ψ-classes intersections*

$$\sum_{\substack{d_1+\cdots+d_n\geq 0 \\ d_1+\cdots+d_n=3g-3+n}} \left(\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{d_i}\right) \prod_{i=1}^n \frac{(2d_i+1)!!}{z_i^{2d_i+2}};$$

(3) *the functions*

$$\frac{\omega_{g,n}^{\mathrm{WK}}(z_1,\ldots,z_n)}{\mathrm{d}z_1\cdots\mathrm{d}z_n}$$

where $\omega_{g,n}^{\mathrm{WK}}$ are the TR amplitudes for the spectral curve

$$S=\mathbb{P}^1, \qquad x(z)=\frac{z^2}{2}, \qquad y(z)=-z, \qquad \omega_{0,1}=y\mathrm{d}x, \qquad \omega_{0,2}(z_1,z_2)=\frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2}. \tag{88}$$

Proof. This identity between (2) and (3) is one of the equivalent form of the Virasoro constraints conjectured in [54] and proved in [36], and it has a long history. It was first obtained in this form in [38], see also [19, 21]. The equivalence between (1) and (2) is straightforward from Corollary 12.2. Here, we want to give a direct proof of the equivalence between (1) and (3). We already know from Corollary 12.2 that $\mathsf{V}_{g,n,\infty}\in\mathbb{Q}[L_1^2,\ldots,L_n^2]$. Therefore, its Laplace transform (87)

$$\mathsf{w}_{g,n}(z_1,\ldots,z_n)=\int_{\mathbb{R}_{>0}^n}\prod_{i=1}^n \mathrm{d}L_i\,L_i\,e^{-z_iL_i}\,\mathsf{V}_{g,n,\infty}(L_1,\ldots,L_n)=(-1)^n\partial_{z_1}\cdots\partial_{z_n}\widehat{\mathsf{V}}_{g,n,\infty}(z_1,\ldots,z_n)$$

is a rational function of $z_1^2,\ldots,z_n^2$ with poles only at $z_i=0$.

We apply Lemma 12.3 to functions f_1 and f_2 in this class, and compute for $\mathrm{Re}\,z_i>0$

$$\begin{aligned}
&\int_{\mathbb{R}_{>0}^n}\prod_{i=1}^n \mathrm{d}L_i\,L_i\,e^{-z_iL_i}\int_{\mathbb{R}_{>0}}\mathrm{d}\ell\,\ell\,\underline{\mathsf{B}}_\infty(L_1,L_2,\ell)\,\mathsf{V}_{g,n-1,\infty}(\ell,L_3,\ldots,L_n)\\
=\;&\partial_{z_2}\Bigg\{\left(-\frac{1}{2z_1^2(z_1+z_2)}+\frac{1}{2z_1^2(z_1-z_2)}\right)\mathsf{w}_{g,n-1}(z_1,z_3,\ldots,z_n)\\
&\quad+\left(\frac{1}{2z_2^2(z_1+z_2)}-\frac{1}{2z_2^2(z_1-z_2)}\right)\mathsf{w}_{g,n-1}(z_2,z_3,\ldots,z_n)\Bigg\}\\
=\;&\partial_{z_2}\left(\frac{z_2^2\mathsf{w}_{g,n-1}(z_1,z_3,\ldots,z_n)-z_1^2\mathsf{w}_{g,n-1}(z_2,z_3,\ldots,z_n)}{z_1^2z_2(z_1^2-z_2^2)}\right)\\
=\;&\operatorname*{Res}_{z\to0}\frac{\mathrm{d}z}{2}\left(\frac{1}{z_1-z}-\frac{1}{z_1+z}\right)\frac{1}{2z^2}\left(\frac{1}{(z-z_2)^2}+\frac{1}{(z+z_2)^2}\right)\mathsf{w}_{g,n-1}(z,z_3,\ldots,z_n).
\end{aligned}$$

The last line can be seen to coincide with the previous line by moving the contour to pick the residues at $z=\pm z_1,\pm z_2$ and using that $\mathsf{w}_{g,n-1}(z,\ldots)$ is an even function of z. With the notations of (88) and

$$\omega_{g,n}(z_1,\ldots,z_n)=\mathsf{w}_{g,n}(z_1,\ldots,z_n)\prod_{i=1}^n\mathrm{d}z_i$$

we therefore recognise, after multiplication by $\mathrm{d}z_1\cdots\mathrm{d}z_n$,

$$\operatorname*{Res}_{z\to0}\frac{\frac{1}{2}\int_{-z}^{z}\omega_{0,2}(\cdot,z_1)}{\omega_{0,1}(z)-\omega_{0,1}(-z)}\big(\omega_{0,2}(z,z_2)\omega_{g,n-1}(-z,z_3,\ldots,z_n)+\omega_{0,2}(-z,z_2)\omega_{g,n-1}(z,z_3,\ldots,z_n)\big).$$

The last identity in Lemma 12.3 gives, for $2g-2+n>1$

$$\begin{aligned}
&\int_{\mathbb{R}_{>0}^n}\prod_{i=1}^n \mathrm{d}L_i\,L_i\,e^{-z_iL_i}\int_{\mathbb{R}_{>0}^2}\mathrm{d}\ell\,\mathrm{d}\ell'\,\ell\,\ell'\,\tfrac{1}{2}\underline{\mathsf{C}}_\infty(L_1,\ell,\ell')\,\mathsf{V}_{g-1,n+1,\infty}(\ell,\ell',L_2,\ldots,L_n)\\
=\;&\frac{1}{2z_1^2}\,\mathsf{w}_{g-1,n+1}(z_1,z_1,z_2,\ldots,z_n)\\
=\;&\operatorname*{Res}_{z\to0}\frac{\mathrm{d}\,z}{2}\left(\frac{1}{z-z_1}-\frac{1}{z+z_1}\right)\frac{1}{2z^2}\,\mathsf{w}_{g-1,n+1}(z,-z,z_2,\ldots,z_n),
\end{aligned}\tag{89}$$

and we recognise, after multiplication by $\mathrm{d}z_1 \cdots \mathrm{d}z_n$,

$$\operatorname*{Res}_{z\to 0} \frac{\frac{1}{2}\int_{-z}^{z} \omega_{0,2}(\cdot, z_1)}{\omega_{0,1}(z) - \omega_{0,1}(-z)}\, \omega_{g-1,n+1}(z, -z, z_2, \ldots, z_n).$$

The Laplace transform of the other terms in the strict GR formula (50) are treated similarly, and we obtain exactly the residue formula of the topological recursion (34). The identities for the case $(g,n) = (1,1)$ can be checked by hand. ■

12.3 Four generating series

As a preliminary to the next paragraph, we consider four generating series

$$t(u) \in \mathbb{C}[[u]], \qquad T(u) \in u\,\mathbb{C}[[u]], \qquad y(z) \in z\,\mathbb{C}[[z^2]] \quad \text{and} \quad \Upsilon(z) \in \mathbb{C}[[z^2]]$$

which are determined from one another by

$$t(u) = \ln(\Upsilon_0) - \ln(1 - T(u)), \qquad T(u) = \frac{2}{(2\pi)^{\frac{1}{2}} u^{\frac{3}{2}}} \int_0^\infty \frac{y_1 z - y(z)}{y_1}\, e^{-\frac{z^2}{2u}}\, z\,\mathrm{d}z \quad \text{and} \quad y(z) = -\frac{z}{\Upsilon(z)}.$$

This makes sense under the following assumptions on leading coefficients

$$y_1 \neq 0, \qquad \Upsilon_0 \neq 0\,.$$

The second equality should be understood coefficient-wise

$$y(z) = \sum_{k\geq 0} y_{2k+1}\, z^{2k+1} \qquad \Longrightarrow \qquad T(u) = -\sum_{k\geq 1} (2k+1)!!\, y_{2k+1}\, u^k.$$

12.4 Translations of Witten-Kontsevich

Given a polynomial

$$\Upsilon(z) = \sum_{k\geq 0} \Upsilon_{2k}\, z^{2k} \in \mathbb{C}[z^2],$$

we introduce an initial data for strict GR, which depends on a parameter $\beta \geq 1$.

$$\begin{aligned}
\underline{\mathsf{A}}_\beta^{(\Upsilon)}(L_1, L_2, L_3) \quad &:= \quad \Upsilon_0 \\
\underline{\mathsf{B}}_\beta^{(\Upsilon)}(L_1, L_2, \ell) \quad &:= \quad \frac{1}{2\beta L_1}\bigg\{\Upsilon(\partial_{L_1})\mathsf{F}_\beta(\ell - L_1 + L_2) - \Upsilon(\partial_{L_2})\mathsf{F}_\beta(\ell + L_1 - L_2) \\
&\qquad + \frac{\Upsilon(\partial_{L_1}) + \Upsilon(\partial_{L_2})}{2}\,\mathsf{F}_\beta(\ell - L_1 - L_2) - \frac{\Upsilon(\partial_{L_1}) - \Upsilon(\partial_{L_2})}{4(\partial_{L_1} - \partial_{L_2})}\big(\mathsf{F}_\beta''(\ell - L_1)L_1 + \mathsf{F}_\beta''(\ell - L_2)L_2\big)\bigg\} \\
\underline{\mathsf{C}}_\beta^{(\Upsilon)}(L_1, \ell, \ell') \quad &:= \quad \frac{1}{\beta L_1}\,\Upsilon(\partial_{L_1})\,\mathsf{F}_\beta(\ell + \ell' - L_1) \\
\underline{\mathsf{D}}_\beta^{(\Upsilon)}(L_1) \quad &:= \quad \int_0^\infty \mathrm{d}\ell\, \ell\, \underline{\mathsf{C}}_\beta^{(\Upsilon)}(L_1, \ell, \ell) = \Upsilon(\partial_{L_1})\,\underline{\mathsf{D}}_\beta(L_1) = \left(\frac{\pi^2}{6\beta^2} + \frac{L_1^2}{24}\right)\Upsilon_0 + \frac{\Upsilon_2}{12}
\end{aligned}$$

where we have used the value of $\underline{\mathsf{D}}_\beta$ given by (83). We recall that $\mathsf{F}_\beta(x) = 2\ln\big(1 + e^{-\frac{\beta x}{2}}\big)$. Note that

$$\frac{\Upsilon(\partial_{L_1}) - \Upsilon(\partial_{L_2})}{\partial_{L_1} - \partial_{L_2}} = \sum_{k_1,k_2\geq 0} \Upsilon_{k_1+k_2+1}\, \partial_{L_1}^{k_1} \partial_{L_2}^{k_2}$$

is a polynomial in ∂_{L_1} and ∂_{L_2}, with the convention that $\Upsilon_{2k+1} = 0$ for all $k \geq 0$. We denote $\mathsf{V}_{g,n,\beta}^{(\Upsilon)}$ the strict GR amplitudes for this initial data.

We lift them to initial data for GR

$$\begin{aligned} \mathsf{X}^{(\Upsilon)}_{\mathbf{P},\beta}(\sigma) &= \underline{X}^{(\Upsilon)}_{\beta}(\ell_\sigma(\partial P)), & \mathsf{X} \in \{\mathsf{A},\mathsf{B},\mathsf{C}\}, \\ \mathsf{D}^{(\Upsilon)}_{\mathbf{T},\beta}(\sigma) &= \sum_{\gamma \in \hat{S}^\circ_T} \mathsf{C}^{(\Upsilon)}_{\beta}(\ell_\sigma(\partial P_\gamma)). \end{aligned}$$

Lemma 12.5 *Let $\Upsilon(z)$ be a fixed even polynomial. For any $\beta \in [1,+\infty)$, the above initial data for GR is M-bounded, uniformly so when β remains bounded. In particular, $\mathsf{D}^{(\Upsilon)}_{\beta}$ is well-defined.*

Proof. First observe that $\mathsf{B}^{(\Upsilon)}_{\beta}$ and $\mathsf{C}^{(\Upsilon)}_{\beta}$ only involve finitely many derivatives of derivatives of order ≥ 2 of F_β. We already know from Section 12.1 that the terms without derivatives are M-bounded – in fact uniformly so for $\beta \in [1,+\infty]$. The other terms only involve derivatives of F_β of order greater of equal to 2. Therefore, it is enough to show the existence of constants $M_{k,\beta} > 0$ such that, for any $\beta \geq 1$ and $k \geq 0$,

$$\left|\mathsf{F}^{(k+2)}_{\beta}(x)\right| \leq M_{k,\beta} \ln\left(1 + e^{-\frac{x}{2}}\right). \tag{90}$$

If this is granted, on the one hand we can specialise this inequality to $x = \ell - L_1 - L_2$, and on the other hand the specialisation to $x = \ell - L_1 + L_2$ and monotonicity yields

$$\left|\mathsf{F}^{(k+2)}_{\beta}(\ell - L_1 + L_2)\right| \leq M_{k,\beta} \ln\left(1 + e^{\frac{L_1 - L_2 - \ell}{2}}\right) \leq M_{k,\beta} \ln\left(1 + e^{\frac{L_1 + L_2 - \ell}{2}}\right).$$

By exchanging the role of L_1 and L_2 we get the same bound for $\left|\mathsf{F}^{((k+2)}_{\beta}(\ell + L_1 - L_2)\right|$. Therefore all the terms in $\mathsf{B}^{(\Upsilon)}_{\beta}$ and $\mathsf{C}^{(\Upsilon)}_{\beta}$ are M-bounded and the proof would be complete.

To justify (90), we compute

$$\mathsf{F}''_{\beta}(x) = \frac{\beta^2}{8\cosh^2\left(\frac{\beta x}{4}\right)}$$

and by induction, one can prove the existence of universal polynomials p_m and $\tilde{p}_m$ of degree m such that, for any $k \geq 2$

$$\mathsf{F}^{(k+2)}_{\beta}(x) = \frac{\beta^{k+2}}{2^{2k+3}\cosh^{k+2}\left(\frac{\beta x}{4}\right)} \cdot \begin{cases} p_{\frac{k-1}{2}}\left[\cosh^2\left(\frac{\beta x}{4}\right)\right]\sinh\left(\frac{\beta x}{4}\right) & \text{if } k \text{ is odd}, \\ \tilde{p}_{\frac{k}{2}}\left[\cosh^2\left(\frac{\beta x}{4}\right)\right] & \text{if } k \text{ is even}. \end{cases}$$

We deduce the existence of a universal constant $c_k > 0$ such that,

$$\forall x \in \mathbb{R}, \qquad \left|\mathsf{F}^{(k+2)}_{\beta}(x)\right| \leq \frac{\beta^{k+2}\, c_k}{4\cosh^2\left(\frac{\beta x}{4}\right)} \leq \beta^{k+2}\, c_k\, e^{-\frac{\beta|x|}{2}} \leq \frac{\beta^{k+2}\, c_k}{\ln 2} \ln\left(1 + e^{-\frac{\beta x}{2}}\right).$$

Applying these inequalities to $x = \ell - L_1 - L_2$, we deduce the inequality (90) we need. ∎

We denote $\Omega^{(\Upsilon)}_{\mathbf{\Sigma},\beta}$ the GR amplitudes for this initial data. We know from Proposition 10.7 that

$$\mathsf{V}^{(\Upsilon)}_{g,n,\beta}(L) = \int_{\mathcal{M}_{\Sigma_{g,n}}(L)} \Omega^{(\Upsilon)}_{\mathbf{\Sigma}_{g,n},\beta} \cdot \nu_{\Sigma_{g,n}}.$$

We introduce the generating series $t(u)$, $T(u)$ and $y(z)$ determined by $\Upsilon(z)$ as described in Section 12.3, with obvious notations for their coefficients. The following result compares all these constructions.

Theorem 12.6 *For $2g - 2 + n > 0$, we have equality of the three following quantities*

(1) *the $\beta \to \infty$ limit of the Laplace transform of the strict GR amplitudes*

$$\lim_{\beta\to\infty} \int_{\mathbb{R}_{>0}} \prod_{i=1}^{n} \mathrm{d}L_i\, L_i\, e^{-z_i L_i}\, \mathsf{V}^{(\Upsilon)}_{g,n,\beta}(L_1,\ldots,L_n).$$

(2) *the integral over Deligne-Mumford compactification of the moduli space*

$$\sum_{\substack{d_1,\ldots,d_n\geq 0 \\ d_1+\cdots+d_k\leq 3g-3+n}} \bigg(\int_{\overline{\mathcal{M}}_{g,n}} \exp\Big[\sum_{d\geq 0} t_d \kappa_d\Big] \prod_{i=1}^{n} \psi_i^{d_i}\bigg) \prod_{i=1}^{n} \frac{(2d_i+1)!!}{z_i^{2d_i+2}}.$$

By the definition and combinatorics of κ-classes, it can alternatively be written

$$\sum_{\substack{d_1,\ldots,d_n\geq 0 \\ d_1+\cdots+d_n\leq 3g-3+n}} \sum_{m\geq 0} \frac{t_0^{2g-2+n}}{m!} \sum_{d'_1,\ldots,d'_m\geq 1} \bigg(\int_{\overline{\mathcal{M}}_{g,n+m}} \prod_{i=1}^{n} \psi_i^{d_i} \prod_{j=1}^{m} \psi_{n+j}^{d'_j}\bigg) \prod_{i=1}^{n} \frac{(2d_i+1)!!}{z_i^{2d_i+2}} \prod_{j=1}^{m} T_{d'_j}\,.$$

(3) *the functions*

$$\frac{\omega_{g,n}(z_1,\ldots,z_n)}{\mathrm{d}z_1\cdots\mathrm{d}z_n}$$

where $\omega_{g,n}$ are the TR amplitudes for the spectral curve

$$S=\mathbb{P}^1, \qquad x(z)=\frac{z^2}{2}, \qquad \omega_{0,1}=y(z)\mathrm{d}x(z), \qquad \omega_{0,2}(z_1,z_2)=\frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}.$$

Proof. The equality between (2) and (3) was obtained in [19]. Here we focus on proving the equality between (1) and (3).

We first evaluate the $\beta\to\infty$ limit of the strict GR initial data. Recalling (82), it amounts to replacing $\frac{F_\beta(-x)}{\beta}$ with $\mathsf{G}(x)$. This limit should be understood in the distributional sense, because the function $\mathsf{G}(x)$ is not continuously differentiable at 0. Indeed $\mathsf{G}'(x)$ is the Heaviside distribution, $\mathsf{G}''(x)=\delta(x)$ is the Dirac distribution, etc. The effect of the derivative ∂_L in Laplace variable is obtained *via* integration by parts. If $f(L)$ is a polynomial in L, we have

$$\widehat{f}(z)=\tfrac{f(0)}{z}+O\big(\tfrac{1}{z^2}\big)$$

and

$$\int_0^\infty \mathrm{d}L\, e^{-zL}\, \partial_L f(L) = f(0) - z\widehat{f}(z) = -[z\widehat{f}(z)]_-$$

where $[g(z)]_-$ is the negative part in the Laurent expansion in z. More generally, if q is a polynomial

$$\int_0^\infty \mathrm{d}L\, e^{-zL} q(\partial_L)\, f(L) = [q(-z)\widehat{f}(z)]_-.$$

In intermediate computations, one can omit to write explicitly the terms coming from the boundary in integration by parts, and only remember at the very end to take the negative part of the Laurent expansion.

We find in Lemma 12.3 all the ingredients to compute the Laplace transform of $\underline{\mathsf{C}}^{(\Upsilon)}_\infty$ and of the three first terms in $\mathsf{B}^{(\Upsilon)}_\infty$. Only the last term in $\underline{\mathsf{B}}^{(\Upsilon)}_\infty$ requires a separate evaluation

$$\begin{aligned} & -\int_{\mathbb{R}^2_{>0}} \mathrm{d}L_1\,\mathrm{d}L_2\, L_1\, L_2\, e^{-z_1L_1-z_2L_2}\, \frac{1}{L_1}\, \frac{\Upsilon(\partial_{L_1})-\Upsilon(\partial_{L_2})}{4(\partial_{L_1}-\partial_{L_2})}\big(\delta(\ell-L_1)L_1+\delta(\ell-L_2)L_2\big) \\ = \;& \partial_{z_2}\bigg\{\frac{\Upsilon(z_1)-\Upsilon(z_2)}{4(z_1-z_2)}\bigg(\frac{e^{-z_1\ell}}{z_1^2}+\frac{e^{-z_2\ell}}{z_2^2}\bigg)\bigg\} + \text{boundary terms} \end{aligned}$$

Hence

$$
\begin{aligned}
& \int_{\mathbb{R}^2_{>0}} \mathrm{d}L_1\,\mathrm{d}L_2\,L_1\,L_2\,e^{-z_1L_1-z_2L_2} \int_{\mathbb{R}_{>0}} \mathrm{d}\ell\,\ell\,\underline{\mathsf{B}}_\infty^{(\Upsilon)}(L_1,L_2,\ell)\,f(\ell) \\
= & \int_{\mathbb{R}_{>0}} \frac{1}{2}\partial_{z_2}\bigg\{ -\frac{\Upsilon(z_1)\,e^{-z_1\ell}}{z_1^2(z_1+z_2)} + \frac{\Upsilon(z_2)\,e^{-z_2\ell}}{z_2^2(z_1+z_2)} + \frac{\Upsilon(z_1)+\Upsilon(z_2)}{2(z_1-z_2)}\bigg(\frac{e^{-z_1\ell}}{z_1^2} - \frac{e^{-z_2\ell}}{z_2^2}\bigg) \\
& + \frac{\Upsilon(z_1)-\Upsilon(z_2)}{4(z_1-z_2)}\bigg(\frac{e^{-z_1\ell}}{z_1^2} + \frac{e^{-z_2\ell}}{z_2^2}\bigg)\bigg\} + \text{boundary terms} \\
= & \left[\partial_{z_2}\bigg(\frac{z_2^2\Upsilon(z_1)(-\widehat{f}'(z_1)) - z_1^2\Upsilon(z_2)(-\widehat{f}'(z_2))}{z_1^2z_2(z_1^2-z_2^2)}\bigg)\right]_- \\
= & \operatorname*{Res}_{z\to 0} \frac{\mathrm{d}z}{2}\bigg(\frac{1}{z_1-z} - \frac{1}{z_1+z}\bigg)\frac{\Upsilon(z)}{z^2}\bigg(\frac{1}{(z-z_2)^2} + \frac{1}{(z+z_2)^2}\bigg)(-\widehat{f}'(z))
\end{aligned}
$$

Likewise, if $f(\ell,\ell')$ is a polynomial in ℓ and ℓ', we compute

$$
\begin{aligned}
& \frac{1}{2}\int_{\mathbb{R}^2_{>0}} \mathrm{d}L_1\,L_1\,e^{-z_1L_1} \int_{\mathbb{R}^2_{>0}} \mathrm{d}\ell\,\mathrm{d}\ell'\,\ell\,\ell'\,\underline{\mathsf{C}}_\infty^{(\Upsilon)}(L_1,\ell,\ell')\,f(\ell,\ell') \\
= & \left[\frac{\Upsilon(z_1)}{z_1^2}\partial_1\partial_2\widehat{f}(z_1,z_1)\right]_- \\
= & \operatorname*{Res}_{z\to 0} \frac{\mathrm{d}z}{2}\bigg(\frac{1}{z_1-z} - \frac{1}{z_1+z}\bigg)\frac{\Upsilon(z)}{z^2}\,\partial_1\partial_2\widehat{f}(z,z).
\end{aligned}
$$

Let us introduce

$$
\omega_{g,n}(z_1,\ldots,z_n) = \bigg(\int_{\mathbb{R}_{>0}} \prod_{i=1}^n \mathrm{d}L_i\,L_i\,e^{-z_iL_i}\,\mathsf{V}^{(\Upsilon)}_{g,n,\infty}(L_1,\ldots,L_n)\bigg)\prod_{i=1}^n \mathrm{d}z_i.
$$

Then, the previous computations shows, as in the end of the proof of Proposition 12.4, that the Laplace transform of the strict GR formula (50) is exactly the original topological recursion residue formula (34) for the claimed spectral curve. ■

12.5 Generalisation to arbitrary $\omega_{0,2}$

As before, we fix $\Upsilon(z) \in \mathbb{C}[z^2]$ with $\Upsilon_0 \neq 0$. Let us fix a generating series $\Phi(z_1,z_2) \in \mathbb{C}[z_1^2,z_2^2]^{\mathfrak{S}_2}$. We assume here it is a polynomial in order to avoid discussing convergence issues. A simple way to include formal series in z_1,z_2 with infinitely many terms at the same time is to consider the coefficients of Φ as independent formal variables.

We introduce initial data for strict GR generalising the one of Section 12.4

$$
\begin{aligned}
\underline{\mathsf{A}}^{(\Upsilon,\Phi)}_\beta(L_1,L_2,L_3) &= \mathsf{A}^{(\Upsilon)}_\beta(L_1,L_2,L_3) = \Upsilon_0 \\
\underline{\mathsf{B}}^{(\Upsilon,\Phi)}_{\beta,\eta}(L_1,L_2,\ell) &= \mathsf{B}^{(\Upsilon)}_\beta(L_1,L_2,\ell) + \sum_{k\geq 0}\Phi_{2k,0}\,\frac{\mathsf{F}^{(2k+2)}_\beta(\eta-\ell)}{\beta}\,\mathsf{A}^{(\Upsilon)}_\beta(L_1,L_2,0) \\
\underline{\mathsf{C}}^{(\Upsilon,\Phi)}_{\beta,\eta}(L_1,\ell,\ell') &= \mathsf{C}^{(\Upsilon)}_\beta(L_1,\ell,\ell') \\
& \quad + \sum_{j,k\geq 0}\Phi_{2j,2k}\bigg(\frac{\mathsf{F}^{(2j+2)}_\beta(\eta-\ell')}{\beta}\,\partial^{(2k)}_{L_2}\underline{\mathsf{B}}^{(\Upsilon)}_\beta(L_1,0,\ell) + \frac{\mathsf{F}^{(2j+2)}_\beta(\eta-\ell)}{\beta}\,\partial^{2k}_{L_2}\underline{\mathsf{B}}^{(\Upsilon)}_\beta(L_1,0,\ell')\bigg) \\
& \quad + \sum_{j,k\geq 0}\Phi_{2j,0}\,\Phi_{2k,0}\,\frac{\mathsf{F}^{(2j+2)}_\beta(\eta-\ell)}{\beta}\,\frac{\mathsf{F}^{(2k+2)}_\beta(\eta-\ell')}{\beta}\,\underline{\mathsf{A}}^{(\Upsilon)}_\beta(L_1,0,0)
\end{aligned}
$$

and induce D *via* (19). Here $\eta > 0$ is a parameter used for regularisation, and will later be sent to 0.

We lift them to initial data for GR

$$\begin{array}{rcll} \mathsf{X}^{(\Upsilon,\Phi)}_{\mathbf{P},\beta,\eta}(\sigma) & = & \underline{X}^{(\Upsilon,\Phi)}_{\beta,\eta}(\ell_\sigma(\partial P)), & \mathsf{X}\in\{\mathsf{A},\mathsf{B},\mathsf{C}\} \\ \mathsf{D}^{(\Upsilon,\Phi)}_{\mathbf{T},\beta,\eta}(\sigma) & = & \sum_{\gamma\in\hat{S}^\circ_T} \mathsf{C}^{(\Upsilon)}_\beta(\ell_\sigma(\partial P_\gamma)). & \end{array}$$

Lemma 12.7 *For any $\beta\in[1,+\infty)$ and $\eta\geq 0$, this initial data is M-bounded, and uniformly so when β remains bounded and $\eta\in[0,1]$.*

Proof. Given Lemma 12.5, we only have to prove that the extra terms involving Φ are M-bounded. From the proof of Lemma 12.5 we know there exists universal constants $c_k>0$ such that, for any $k\geq 0$, $x\in\mathbb{R}$ and $L_1,L_2,\ell\geq 0$,

$$\begin{array}{rcl} \left|\mathsf{F}^{(2k+2)}_\beta(x)\right| & \leq & \beta^{k+2}\,c_k\,e^{-\frac{\beta|x|}{2}} \leq \beta^{k+2}\,c_k\,e^{-\frac{|x|}{2}}, \\ \left|\partial^{2k}_{L_2}\mathsf{B}^{(\Upsilon)}_\beta(L_1,L_2,\ell)\right| & \leq & M_{\beta,k}\,\ln\left(1+e^{\frac{L_1-\ell}{2}}\right) \end{array}$$

where $M_{\beta,k}$ is uniform when β and k remain bounded. With $x=\ell'-\eta$, using the inequality

$$\forall a>0,\qquad \forall b<1,\qquad \frac{b\,\ln(1+a)}{\ln(1+ab)}\leq 1 \tag{91}$$

with $a=e^{\frac{L_1-\ell}{2}}$ and $b=e^{-\frac{\ell'}{2}}$, we deduce that

$$\left|\mathsf{F}^{(2j+2)}_\beta(\eta-\ell')\,\partial^{2k}_{L_2}\mathsf{B}^{(\Upsilon)}_\beta(L_1,0,\ell)\right|\leq M_{\beta,k}\,\beta^{j+2}\,c_j\,\ln\left(1+e^{\frac{L_1-\ell}{2}}\right)e^{-\frac{|\ell'-\eta|}{2}}\leq M_{\beta,k}\,\beta^{j+2}\,c_j\,\ln\left(1+q(\eta,\ell')\,e^{\frac{L_1-\ell-\ell'}{2}}\right)$$

where $q(\eta,\ell')=e^{\frac{\ell'-|\ell'-\eta|}{2}}$. We also have

$$\forall\eta\in[0,1],\qquad \forall\ell'\in\mathbb{R}_{\geq 0},\qquad q(\eta,\ell')\leq e^{\frac{1}{2}}$$

and using again the inequality (91) with $a=q(\eta,\ell')e^{\frac{L_1-L_2-\ell}{2}}$ and $b=q^{-1}(\eta,\ell')$ one gets

$$\left|\mathsf{F}^{(2j+2)}_\beta(\eta-\ell')\,\partial^{2k}_{L_2}\mathsf{B}^{(\Upsilon)}_\beta(L_1,0,\ell)\right|\leq M_{\beta,k}\,\beta^{j+2}\,c_j\,e^{\frac{1}{2}}\,\ln\left(1+e^{\frac{L_1-\ell-\ell'}{2}}\right).$$

The other terms in $\mathsf{B}^{(\Upsilon,\Phi)}_{\beta,\eta}$ and $\mathsf{C}^{(\Upsilon,\Phi)}_{\beta,\eta}$ are treated similarly, and yield the conclusion. ∎

We denote $\Omega^{(\Upsilon)}_{\mathbf{\Sigma},\beta}$ (respectively $\mathsf{V}^{(\Upsilon,\Phi)}_{g,n,\beta,\eta}$) the (strict) GR amplitudes for this initial data. We know from Proposition 10.7 that

$$\mathsf{V}^{(\Upsilon,\Phi)}_{g,n,\beta,\eta}(L)=\int_{\mathcal{M}_{\Sigma_{g,n}}(L)}\Omega^{(\Upsilon,\Phi)}_{\mathbf{\Sigma}_{g,n},\beta,\eta}\cdot\nu_{\Sigma_{g,n}}.$$

Let $\mathrm{Div}_{g,n}$ be the set of boundary divisors of $\overline{\mathcal{M}}_{g,n}$. If $\mathsf{d}\in\mathrm{Div}_{g,n}$, we denote $i_{\mathsf{d}}:\partial\overline{\mathcal{M}}_{g,n}\to\overline{\mathcal{M}}_{g,n}$ the natural inclusion, and $\psi_\circ,\psi_\bullet$ the first Chern class of the cotangent line bundle attached to the two nodes appearing on the divisor d. We introduce the formal combination of these classes

$$[\Phi]:=\exp\left(\frac{1}{2}\sum_{\substack{k,l\geq 0\\ \mathsf{d}\in\mathrm{Div}_{g,n}}}(2k-1)!!\,(2l-1)!!\,\Phi_{2k,2l}\,i_{\mathsf{d}*}\left(\psi_\circ^k\psi_\bullet^l\right)\right)$$

where g and n are specified by the context.

Theorem 12.8 *For $2g-2+n>0$, we have equality of the three following quantities*

(1) *the (limit of) of the Laplace transform of the strict GR amplitudes*

$$\lim_{\eta\to 0^+}\lim_{\beta\to\infty}\int_{\mathbb{R}_{>0}}\prod_{i=1}^{n}\mathrm{d}L_i\,L_i\,e^{-z_iL_i}\,\mathsf{V}^{(\Upsilon,\Phi)}_{g,n,\beta,\eta}(L_1,\ldots,L_n)$$

(2) *the integrals over Deligne-Mumford compactification of the moduli space*

$$\sum_{\substack{d_1,\ldots,d_n\geq 0\\ d_1+\cdots+d_k\leq 3g-3+n}}\bigg(\int_{\overline{\mathcal{M}}_{g,n}}[\Phi]\,e^{\sum_{d\geq 0}t_d\kappa_d}\prod_{i=1}^{n}\psi_i^{d_i}\bigg)\prod_{i=1}^{n}\frac{(2d_i+1)!!}{z_i^{2d_i+2}}$$

(3) *the functions*

$$\operatorname*{Res}_{z_1'\to 0}\cdots\operatorname*{Res}_{z_n'\to 0}\frac{\omega_{g,n}(z_1',\ldots,z_n')}{\prod_{i=1}^{n}(z_i-z_i')} \tag{92}$$

where $\omega_{g,n}$ are the TR amplitudes for the spectral curve

$$S=\mathbb{P}^1,\qquad x(z)=\frac{z^2}{2},\qquad \omega_{0,1}=y(z)\mathrm{d}x(z)\ +\ \text{non odd}$$

and

$$\omega_{0,2}(z_1,z_2)=\frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}+\sum_{k_1,k_2\geq 0}\Phi_{2k_1,2k_2}\,z_1^{2k_1}\,z_2^{2k_2}\mathrm{d}z_1\mathrm{d}z_2\ +\ \text{non odd}$$

where the non-odd (separately in each variable) parts in z_i are arbitrary. Note that (92) extracts the divergent part of the $\omega_{g,n}(z_1,\ldots,z_n)$.

Proof. Again, the equality between (2) and (3) was obtained in [20]. We only sketch the proof of the equality between (1) and (3), as many computations are similar to the proof in Theorem 12.6. The limit $\beta\to\infty$ of the strict GR initial data we propose exists at the distribution level

$$\begin{aligned}
\underline{\mathsf{B}}^{(\Upsilon,\Phi)}_{\infty,\eta}(L_1,L_2,\ell) &= \underline{\mathsf{B}}^{(\Upsilon)}_{\infty}(L_1,L_2,\ell)+\sum_{k\geq 0}\delta^{(2k)}(\ell-\eta)\,\underline{\mathsf{A}}^{(\Upsilon)}_{\beta}(L_1,L_2,0)\\
\underline{\mathsf{C}}^{(\Upsilon,\Phi)}_{\infty,\eta}(L_1,\ell,\ell') &= \underline{\mathsf{C}}^{(\Upsilon)}_{\infty}(L_1,\ell,\ell')+\sum_{j,k\geq 0}\Phi_{2j,2k}\big(\delta^{(2j)}(\ell'-\eta)\,\partial_{L_2}^{2k}\underline{\mathsf{B}}^{(\Upsilon)}_{\infty}(L_1,0,\ell)+\delta^{(2j)}(\ell-\eta)\,\partial_{L_2}^{2k}\underline{\mathsf{B}}_{\infty}(L_1,0,\ell')\big)\\
&\quad+\sum_{j,k\geq 0}\Phi_{2j,0}\,\Phi_{2k,0}\,\delta^{(2j)}(\ell-\eta)\,\delta^{(2k)}(\ell'-\eta)\,\underline{\mathsf{A}}^{(\Upsilon)}_{\infty}(L_1,0,0)
\end{aligned}$$

When we compute the integral

$$\int_{\mathbb{R}_{>0}}\mathrm{d}\ell\,\ell\,\underline{\mathsf{B}}^{(\Upsilon,\Phi)}_{\infty,\eta}(L_1,L_2,\ell)\,f(\ell)$$

the effect of $\delta^{(2k)}(\ell-\eta)$ is to evaluate the derivative of $f(\ell)$ at $\ell=\eta$. Taking the limit $\eta\to 0^+$ therefore amounts to take the derivative[5] at $\eta=0$. In Laplace variable, it amounts to multiply by z^{2k} up to boundary terms.

On the other hand, in the topological recursion, the replacement of $\omega_{0,2}$

$$\frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}\longrightarrow\frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}+\Phi(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2$$

amounts in the notation of Section 7.4 to replacing the 1-forms ξ_k

$$\frac{(2k+1)\,\mathrm{d}z}{z^{2k+2}}\longrightarrow\frac{(2k+1)\,\mathrm{d}z}{z^{2k+2}}+\sum_{l\geq 0}\Phi_{2k,2l}\,z^{2l}\,\mathrm{d}z. \tag{93}$$

[5] It was necessary to introduce η as the integration range is only the half-real line, so simply taking the $\beta\to\infty$ of $\beta^{-1}\mathsf{F}_{\beta}^{(2k+2)}(-\ell)$ would rather produce a half of the desired result.

It has for effect to replace the coefficients of the Airy structure $(A,B,C) \to (\tilde{A},\tilde{B},\tilde{C})$ with

$$\begin{cases} \tilde{A}^i_{j,k} & = & A^i_{j,k} \\ \tilde{B}^i_{j,k} & = & B^i_{j,k} + \sum_{a\geq 0} \Phi_{2k,2a} A^i_{j,a} \\ \tilde{C}^i_{j,k} & = & C^i_{j,k} + \sum_{a\geq 0} \Phi_{2k,2a}\, B^i_{a,j} + \Phi_{2j,2a}\, B^i_{a,k} + \sum_{a,b\geq 0} \Phi_{2j,2a}\, \Phi_{2k,2b}\, A^i_{a,b} \end{cases}$$

Recalling that

$$\int_{\mathbb{R}_{>0}} \mathrm{d}L\, L\, e^{-zL}\, L^{2k} = \frac{(2k+1)!}{z^{2k+2}}, \qquad \partial_L^{2k}(L^{2k}) = (2k)! = \frac{(2k+1)!}{2k+1}$$

and transporting the modifications of $\underline{\mathsf{B}}$ and $\underline{\mathsf{C}}$ induced by Φ in Laplace transform, one identifies with (93)-(94) the additional terms in the strict GR initial data. The identity relating $\underline{\mathsf{D}}$ to the topological recursion's $\omega_{1,1}$ can be checked separately. ■

12.6 General spectral curves

For spectral curves possibly having several simple branchpoints, a similar theorem can be formulated and proved, by considering the target

$$\mathbf{E}(\mathbf{\Sigma}) = \mathcal{C}^0(\mathcal{T}_\Sigma, \mathcal{A})$$

where $\mathcal{A}$ is the Frobenius algebra given by the Jacobi ring, equipped with its canonical basis given by the branchpoints. Now the initial data (A,B,C) – respectively D – will be valued in $\mathcal{A}^* \otimes \mathcal{A}^{\otimes 2}$ – respectively $\mathcal{A}^*$ – and we use the product of functions which appear all over our formulas now also involve the product in the Frobenius algebra, and the pairing to identify $\mathcal{A} \cong \mathcal{A}^*$.

In many application of the topological recursion, one need to consider $\Upsilon(z)$ and $\Phi(z_1,z_2)$ which are formal series with complex coefficients, rather than just polynomials. In doing so, one should then address the question of convergence of the series which define the corresponding (strict) GR initial data. There should be however be no surprise, in the sense that once the series are guaranteed to be convergent the result of Theorems 12.6 and 12.8 continue to hold.

12.7 Comments

Interestingly, the GR amplitudes $\Omega_{\mathbf{\Sigma}}$ we have described in the previous paragraphs give actual functions on the moduli space $\mathcal{M}_\Sigma(L)$, whose integration results in intersection numbers of ψ-cohomology classes, and this formula was derived without constructing differential form representatives for these cohomology classes. Even in the simplest case of the Witten-Kontsevich model, we do not know of a simple expression for $\Omega_{\mathbf{\Sigma},\beta}$ and $\Omega_{\mathbf{\Sigma},\infty}$ other than their definition by the geometric recursion. Therefore, the lift from TR amplitudes to GR amplitudes, which we have described for an arbitrary spectral curve, seems rather non-trivial.

Our results apply in particular to cohomological field theory $\mathcal{H}$ on a semi-simple Frobenius algebra $\mathcal{A}$, modulo solving the convergence question for initial data raised in Section 12.6 for the examples one is interested in – see (94) below. Indeed, Teleman [49] has proved that the action of the Givental group [31, 32] on cohomological field theories on $\mathcal{A}$ is transitive. Therefore,

$$\mathcal{H} \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes \mathcal{A}^{\otimes n}$$

can be obtained by action on the trivial cohomological field theory

$$\bigoplus_a [1]_{\overline{\mathcal{M}}_{g,n}} \otimes \mathbf{1}^{\otimes n} \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes \mathcal{A}^{\otimes n}$$

whose correlation functions are captured by a sum of Witten-Kontsevich models. The Givental group element is described by a generating series $R(u) \in (\mathrm{End}\,\mathcal{A})[[u]]$ such that $R(u)R^{\dagger}(-u) = \mathrm{Id}$. It specifies two generating series

$$T(u) = u(\mathrm{Id} - R(u)) \cdot \mathbf{1} \qquad \widetilde{\Phi}(u_1, u_2) = \frac{b^{\dagger} - R(u_1) \otimes R(u_2) \cdot b^{\dagger}}{u_1 + u_2} \in (\mathcal{A} \otimes \mathcal{A})[[u_1, u_2]]$$

where $\mathbf{1}$ is the unit and $b^{\dagger}$ is the element of $\mathcal{A}^{\otimes 2}$ specified by the pairing. $T(u)$ together with $\Upsilon_0 = \mathbf{1}$ determines a generating series $\Upsilon(z) \in \mathcal{A}[[z^2]]$ as in Section 12.3, and $\widetilde{\Phi}$ determines a generating series $\Phi(z_1, z_2) \in (\mathcal{A} \otimes \mathcal{A})[[z_1, z_2]]$ *via*

$$\widetilde{\Phi}(u_1, u_2) = \frac{2}{\pi (u_1 u_2)^{\frac{3}{2}}} \int_{\mathbb{R}^2_{>0}} \mathrm{d}z_1\, \mathrm{d}z_2\, z_1\, z_2\, e^{-\frac{z_1^2}{2u_1} - \frac{z_2^2}{2u_2}}\, \Phi(z_1, z_2) \qquad \Longleftrightarrow \qquad \Phi_{2k,2l} = \widetilde{\Phi}_{k,l}(2k-1)!!(2l-1)!! \tag{94}$$

[17] proved that for $2g - 2 + n > 0$, the correlation functions of the cohomological field theory

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{\substack{d_1, \ldots, d_n \geq 0 \\ d_1 + \cdots + d_n \leq 3g-3+n}} \left(\int_{\overline{\mathcal{M}}_{g,n}} \mathcal{H} \prod_{i=1}^{n} \psi_i^{d_i} \right) \cdot \bigotimes_{i=1}^{n} \left[\frac{(2d_i+1)!!\, \mathbf{1}\, \mathrm{d}z_i}{z_i^{2d_i+2}} + (2d_i - 1)!! \sum_{l_i \geq 0}^{\infty} \Phi_{2d_i, l_i} z_i^{l_i} \mathrm{d}z_i \right] \tag{95}$$

are then computed by the topological recursion for the local spectral curve consisting of several copies of the formal neighbourhood of $0 \subset \mathbb{C}$ equipped with the branched cover $x(z) = \frac{z^2}{2}$, with $\omega_{0,1}$ and $\omega_{0,2}$ given as in Theorem 12.8. It is understood in (95) we multiply $\mathcal{H}$ with the $\mathcal{A}^{\otimes n}$-valued multidifferential to its right, factorwise and using the product in the Frobenius algebra. We refer to [3, Section 4] for a precise statement close to our notations. Our results show that one can approximate (95) by the Laplace transform of the integral over $\overline{\mathcal{M}}_{\Sigma_{g,n}(L)}$ of GR amplitudes against Weil-Petersson volumes.

The correspondence becomes exact at $\beta = \infty$. However, beyond the Witten-Kontsevich model, the initial data for $\beta = \infty$ involve distributions. Introducing $\beta \geq 1$ finite allows their regularisation, so that $\Omega_{\mathbf{\Sigma},\beta}$ are continuous functions on the Teichmüller spaces. In principle, one can imagine to construct target theories from spaces of distributions of Teichmüller space and make sense of these GR initial data and amplitudes at $\beta = \infty$. Doing so precisely brings technicalities which we leave out of the scope of the present article. We can already point two difficulties which should be addressed to achieve this.

First, if we have a smooth function over $\mathcal{T}_{\Sigma}$, its derivatives with respect to boundary lengths are not unambiguously defined, because the fibration $L : \mathcal{T}_{\Sigma} \to \mathbb{R}^n_{>0}$ does not carry a natural Ehresmann connection. In fact, the structure of our computations suggests an *ad hoc* solution: when we need to differentiate with respect to the length of a boundary component b, we are always in the situation where we dispose of an embedded pair of pants P_{γ} in Σ which bounds b. Then, we have a canonical deformation of the hyperbolic metric on P_{γ} which varies $\ell_{\sigma}(b)$ while keeping the lengths of the other boundary components of P_{γ} fixed. In other words, we can use the well-defined vector field ∂_{L_b} on $\mathcal{T}_{P_{\gamma}} \cong \mathbb{R}^3_{>0}$ and

(1) Lift it uniquely to the field on $\widetilde{\mathcal{T}}_{\mathbf{P}_{\gamma}}$ which is invariant under $U(1)^3$ acting transitively on the fiber of $\widetilde{\mathcal{T}}_{\mathbf{P}_{\gamma}} \to \mathcal{T}_P$;

(2) Extend it to a vector field on $\widetilde{\mathcal{T}}_{\mathbf{P}_{\gamma}} \times \widetilde{\mathcal{T}}_{\mathbf{\Sigma}_{\gamma}}$ by requiring its projection on the tangent space of the second factor is 0;

(3) Restrict it to the locus $\big(\widetilde{\mathcal{T}}_{\mathbf{P}_{\gamma}} \times \widetilde{\mathcal{T}}_{\mathbf{\Sigma}_{\gamma}}\big)^{=}$ where lengths of curves we glue on match;

(4) Push it forward via the glueing map $\tilde{\vartheta}_{\gamma}$ to obtain a well-defined vector field on $\widehat{\mathcal{T}}_{\mathbf{\Sigma}}$. This is enough if we want to work over Teichmüller space of pointed bordered surfaces;

(5) If we want to work over the Teichmüller space of bordered surfaces, we notice that the outcome of the previous step is invariant under $U(1)^n$ acting on the fibers of $\widetilde{\mathcal{T}}_{\mathbf{\Sigma}} \to \mathcal{T}_{\Sigma}$, therefore is the lift of a unique vector field on $\mathcal{T}_{\Sigma}$, which we may denote $\partial_{L_b}^{(\gamma)}$.

A second problem appears for the setting of Section 12.5, as Φ couples to the Taylor expansion at 0 boundary lengths of the function we glue the initial data to. Therefore, in order to run GR, one would need to control smoothness up to zero boundary lengths and existence of this Taylor expansion, in the spirit of Sections 9.6 but for boundary components and including derivative controls.

13 Coupling to conformal field theories

In this section, we describe the target theory given by the space of continuous functions from Teichmüller space to the space of states of certain two-dimensional conformal field theories (2d CFT). The notion of 2d CFT we work with is axiomatised by modular functors. We first need the notion of labelled and decorated surfaces.

13.1 Decorated bordered surfaces

This is a slight refinement of the category of pointed bordered surfaces, that occurs in TQFTs with anomalies.

Definition 13.1 *A decorated bordered surface is an ordered pair* $(\mathbf{\Sigma}, \mathcal{L})$*, where* $\mathbf{\Sigma}$ *is a pointed bordered surface and* $\mathcal{L}$ *is a Lagrangian sublattice of* $H_1(\Sigma, \partial\Sigma; \mathbb{Z}) \coloneqq H_1(\Sigma, \mathbb{Z})/i_* H_1(\partial\Sigma, \mathbb{Z})$ *where* $i : \partial\Sigma \to \Sigma$ *is the inclusion.*

We form the category of decorated bordered surfaces $\widetilde{\mathrm{Bord}}^{\bullet}$ whose objects are decorated bordered surfaces and whose morphisms are pairs (f, ς), where $f : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$ is a morphism of pointed bordered surfaces, and ς is a homotopy class of paths from $f_*\mathcal{L}_1$ to $\mathcal{L}_2$ in the space $\{\text{Lagrangian subspace of } H_1(\Sigma_2, \mathbb{R})/i_* H_1(\Sigma_1, \mathbb{R})\}$. In this case, the composition of morphisms is obvious.

One can use an equivalent description, where one takes as morphisms ordered pairs (f, s) where $s \in \mathbb{Z}$, but the composition of two morphisms $(f_1, s_1) : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$ and $(f_2, s_2) : \mathbf{\Sigma}_2 \to \mathbf{\Sigma}_3$ is defined as follows

$$(f_2, s_2) \circ (f_1, s_1) = (f_2 \circ f_1, s_1 + s_2 - \sigma(f_{2*} f_{1*} L_1, f_{2*} L_2, L_3)),$$

where σ is the Maslov index of the three lagrangians [52].

Centrally extended mapping class group

The automorphism group of an object $(\mathbf{\Sigma}, \mathcal{L})$ in $\widetilde{\mathrm{Bord}}^{\bullet}$ is denoted $\widetilde{\Gamma}_{\mathbf{\Sigma}}$, and it is a central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\Gamma}_{\mathbf{\Sigma}} \longrightarrow \Gamma_{\mathbf{\Sigma}} \longrightarrow 1. \tag{96}$$

Glueing operations

There exist glueing operations on decorated bordered surfaces. Let $(\mathbf{\Sigma}, \mathcal{L})$ be a decorated bordered surface, and select an ordered pair $\gamma = (b', b'')$ of boundary components. Choose arbitrarily an orientation-reversing (with respect to the induced orientations from Σ) diffeomorphism $\varphi : (\partial_{b'}\Sigma, o_{b'}) \to (\partial_{b''}\Sigma, o_{b''})$. We already know how to construct another pointed bordered surface $\Sigma_\gamma = \Sigma/\sim$, where $\sim$ is the equivalence relation generated by φ. The projection gives a continuous map $p : \Sigma \to \Sigma_\gamma$. We construct the lagrangian sublattice $\mathcal{L}_b$ as follows. Let Σ' the topological space obtained from Σ by identifying $\partial_{b'}\Sigma \cup \partial_{b''}\Sigma$ to a point. We have continuous maps $q : \Sigma_\gamma \to \Sigma'$, and $n : \Sigma \to \Sigma'$. On the first homology group relative boundary, n induces an injection and q a surjection, so we can define $\mathcal{L}_\gamma = q_*^{-1}(n_*(\mathcal{L}))$. Note that $p_*\mathcal{L}$ is contained in $\mathcal{L}_\gamma$, and that the homology class of the (common) image of $\partial_{b'}\Sigma$ (and $\partial_{b''}\Sigma$) in Σ_γ is also in $\mathcal{L}_\gamma$.

As before, a different choice of φ yields a canonically up to isotopy diffeomorphic glued surface. This construction also means there is an operation of cutting a decorated bordered surface along any pointed oriented multicurve (γ, o) in Σ°, provided the homology classes of the connected components of γ belongs to $\mathcal{L}$.

The centrally extended mapping class groups $\widetilde{\Gamma}_{\mathbf{\Sigma}_b}$ and $\widetilde{\Gamma}_{\mathbf{\Sigma}^\gamma}$ are related by an exact sequence similar to (3).

13.2 Labelled (decorated) surfaces

Definition 13.2 *A label set is a set Λ together an involution $\dagger : \Lambda \to \Lambda$. A pointed label set is a label set together with a distinguished element $1 \in \Lambda$ such that $1^\dagger = 1$.*

We will encounter another category $\widetilde{\mathrm{Bord}}^\bullet(\Lambda)$. Its objects are triples $(\mathbf{\Sigma}, \mathcal{L}, \boldsymbol{\lambda})$, where $(\mathbf{\Sigma}, \mathcal{L})$ is a decorated bordered surface, and $\boldsymbol{\lambda}$ is a "labelling" map from $\pi_0(\partial\Sigma)$ to Λ, and morphisms of Λ-decorated bordered surfaces are just morphisms of decorated bordered surfaces preserving labellings.

In this category, we only allow glueing operations which identify boundary components b and b' such that $\boldsymbol{\lambda}(b) = (\boldsymbol{\lambda}(b'))^\dagger$ – and this piece of information is forgotten in the glued surface. Likewise, we can specify a cutting operation for any pointed oriented multicurve γ together with a map $\pi_0(\gamma) \to \Lambda$.

Dropping the data of a Lagrangian sublattice, we can form the category $\mathrm{Bord}^\bullet(\Lambda)$ whose objects consist of a pointed bordered surface $\mathbf{\Sigma}$ together with $\boldsymbol{\lambda} : \pi_0(\partial\Sigma) \to \Lambda$ as above.

13.3 Modular functors

A modular functor is an axiomatic way to specify a two-dimensional conformal field theory. This notion can be traced back to [43, 46, 51]. Several variants of this notion, which all bear the name "modular functor", have been used since then in the literature. Here we adopt the definition spelled out in [5], which does not assume modular functors come from a modular tensor category. In brief, a *modular functor* is the data of a finite pointed label set $(\Lambda, \dagger, 1)$ together with a monoidal functor $\mathcal{Z}$ from $(\widetilde{\mathrm{Bord}}^\bullet(\Lambda), \sqcup)$ to the category $\mathrm{Vect}_\mathbb{C}$ of finite-dimensional complex vector spaces with $\otimes$ as monoidal structure, as well as glueing isomorphisms satisfying a factorisation axiom

$$\Psi_\gamma : \mathcal{Z}(\mathbf{\Sigma}_b, \boldsymbol{\lambda}) \xrightarrow{\sim} \bigoplus_{\mu \in \Lambda} \mathcal{Z}\big(\mathbf{\Sigma}^\gamma, (\boldsymbol{\lambda}, \mu, \mu^\dagger)\big)$$

where μ is the label of b' and $\mu^\dagger$ the label of b'', and canonical isomorphisms

$$\mathcal{Z}(\mathrm{disk}, \lambda) \cong \begin{cases} \mathbb{C} & \text{if } \lambda = 1 \\ 0 & \text{otherwise} \end{cases}, \qquad \mathcal{Z}(\mathrm{cylinder}, \lambda, \mu) \cong \begin{cases} \mathbb{C} & \text{if } \lambda = \mu^\dagger \\ 0 & \text{otherwise} \end{cases}$$

which all satisfy natural compatibility conditions. Note that the source category in [5] is rather a category of "marked surfaces". It is however clear by capping the boundary of our surfaces with punctured disks that

Lemma 13.3 *The category of decorated bordered surfaces $\widetilde{\mathrm{Bord}}^{\bullet}$ together with the glueing operations described above, is equivalent to the category of marked surfaces with its glueing operations described in [5].*

Physically, Λ is the set of boundary conditions, and $\mathcal{Z}(\mathbf{\Sigma}, \boldsymbol{\lambda})$ is the space of states of the CFT on the surface $\mathbf{\Sigma}$ with boundary conditions $\boldsymbol{\lambda}$.

As a consequence of the factorisation axioms, the generator of $\mathbb{Z}$ in the description of $\tilde{\Gamma}_{\mathbf{\Sigma}}$ by the exact sequence (96) acts by a scalar on $\mathcal{Z}(\mathbf{\Sigma})$, which is independent of $\mathbf{\Sigma}$ and denoted $\tilde{c}$. Likewise, the Dehn twist around a boundary component β of $\mathbf{\Sigma}$ acts by a scalar on $\mathcal{Z}(\mathbf{\Sigma})$, which only depends on the label $\lambda(\beta)$ and not on the topology of $\mathbf{\Sigma}$, and is denoted $\tilde{r}_{\lambda(\beta)}$.

Definition 13.4 *We say that the modular functor has zero central charge when $\tilde{c} = 1$, and that it has zero conformal dimensions if $\tilde{r}_{\lambda} = 1$ for all $\lambda \in \Lambda$.*

Besides, if $\mathbf{\Sigma}, \mathbf{\Sigma}'$ are objects of $\widetilde{\mathrm{Bord}}^{\bullet}(\Lambda)$ such that $\mathbf{\Sigma}'$ differs from $\mathbf{\Sigma}$ by addition of marked points labelled by 1, a modular functor provides canonical isomorphisms $\mathcal{Z}(\mathbf{\Sigma}) \cong \mathcal{Z}(\mathbf{\Sigma}')$. If the modular functor has central charge zero, it drops to a functor from $\mathrm{Bord}^{\bullet}(\Lambda)$ to $\mathrm{Vect}_{\mathbb{C}}$ that we still denote $\mathcal{Z}$.

We introduce the weaker notion of *stable modular functor*. It is the data of a finite label set $(\Lambda, \dagger)$ together with a monoidal functor from the full subcategory of $(\widetilde{\mathrm{Bord}}^{\bullet}(\Lambda), \sqcup)$ consisting of stable surfaces, to the category of finite-dimensional vector spaces, that satisfies all the usual axioms of modular functors, except the one specifying initial conditions for disks and cylinders. There is no need to assume the data of a distinguished element $1 \in \Lambda$ in this definition as we do not have to specify the value of the functor on disks. We say that a stable modular functor is *unital* if it is supplemented with the data of a distinguished element $1 \in \Lambda$ such that $1 = 1^{\dagger}$, and with the data, for any ordered pair of objects $\big((\mathbf{\Sigma}, \boldsymbol{\lambda}), (\mathbf{\Sigma}', \boldsymbol{\lambda}')\big)$ that differ from each other by addition of marked points labelled 1, of canonical isomorphisms $\mathcal{Z}(\mathbf{\Sigma}, \boldsymbol{\lambda}) \cong \mathcal{Z}(\mathbf{\Sigma}', \boldsymbol{\lambda}')$ compatible with the union and glueing morphisms.

To allow for infinite-dimensional space of states, we introduce the notion of *stable Hilbert modular functor*[6]. It is the data of a measured space (Λ, ν) with a measure-preserving involution $\dagger$, together with a monoidal functor from the full subcategory of $(\widetilde{\mathrm{Bord}}^{\bullet}(\Lambda), \sqcup)$ consisting of stable surfaces, to the category of separable Hilbert spaces with the completed tensor product as monoidal structure, satisfying axioms similar to the ones defining stable modular functors, and such that the hermitian structure is compatible with union and glueing operations. The only difference is that the direct sum in the glueing axiom is replaced by a direct integral over Λ with respect to ν.

Obviously, a modular functor restricts to a unital stable modular functor, and a stable unitary modular functor is a special case of Hilbert modular functor where Λ is finite and ν is the counting measure. Rational 2d CFTs give rise to modular functors, whereas some nonrational 2d CFTs like quantum Teichmüller theory [50] can give rise to stable Hilbert modular functors.

Tensorial operations

All these notions of modular functors are closed under taking the dual space, and making tensor products. In particular, if $\mathcal{Z}$ is a modular functor of any of the kind above, $(\mathbf{\Sigma}, \boldsymbol{\lambda}) \mapsto \mathrm{End}\,\mathcal{Z}(\mathbf{\Sigma}, \boldsymbol{\lambda})$ is a modular functor of the same kind, but in any case with zero central charge and zero conformal dimensions. In other words, the representations of $\widetilde{\Gamma}_{\mathbf{\Sigma}}$ (see the exact sequence (96)) drops to a representation of Γ_{Σ}.

[6] This is called a stable unitary modular functor in [50].

13.4 Definition of the target theory

Over bordered Teichmüller space

The target theory we are going to construct is an example of the procedure of "fibering over Teichmüller space" described in Section 10.3. Let $\mathcal{Z} : \mathrm{Bord}_1^{\bullet}(\Lambda) \to \mathrm{Vect}_{\mathbb{C}}$ be a stable Hilbert modular functor with zero central charge and zero conformal dimensions. For instance, it could be the endomorphism theory of any stable Hilbert modular functor. We put

$$\mathbf{E}(\mathbf{\Sigma}) = \mathcal{C}^0\big(\mathcal{T}_\Sigma, \mathcal{Z}(\mathbf{\Sigma})\big).$$

Equivalently, this is the space of continuous sections of the trivial vector bundle $\mathcal{T}_\Sigma \times \mathcal{Z}(\mathbf{\Sigma}) \to \mathcal{T}_\Sigma$. This bundle carries an action of Γ_Σ which covers the action of the mapping class group on $\mathcal{T}_\Sigma$.

Let $\|\cdot\|_{\mathbf{\Sigma}}$ the Hilbert norm on $\mathcal{Z}(\mathbf{\Sigma})$, which is functorial according to the axioms. We let $\mathscr{I}_{\mathbf{\Sigma}} = \mathbb{R}_{>0}$ and for any $\varepsilon > 0$, $\mathscr{A}_{\mathbf{\Sigma}}^{\varepsilon}$ is the set of all compacts $K \subset K_\Sigma(\varepsilon)$. We take

$$E^{\varepsilon}(\mathbf{\Sigma}) = \mathcal{C}^0\big(K_\Sigma(\varepsilon), \mathcal{Z}(\mathbf{\Sigma})\big)$$

equipped with the seminorms

$$|f|_K = \sup_{\sigma \in K} \|f(\sigma)\|_{\mathbf{\Sigma}}.$$

The union morphism is defined using the monoidal structure provided by $\mathcal{Z}$

$$(f_1 \sqcup f_2)(\sigma_1, \sigma_2) = f_1(\sigma_1) \otimes f_2(\sigma_2).$$

The glueing morphism is defined using the isomorphisms Ψ given along with $\mathcal{Z}$ and the glueing fibration (40)

$$\mathbf{\Theta}_\gamma(f_1, f_2)(\sigma) = \Psi_\gamma^{-1}((f_1 \sqcup f_2)(\sigma'))$$

for an arbitrary $\sigma' \in \tilde{p}_{\mathbf{\Sigma}_\gamma}(\vartheta^{-1}(\tilde{p}_{\mathbf{\Sigma}}^{-1}(\sigma)))$. The length functions are defined from hyperbolic lengths as in Section 9.1.

The assumption that the central charge and the conformal dimensions vanish allows to work over the Teichmüller space instead of the total space of some circle bundles over Teichmüller space. The assumption of a Hilbert (or unitary) structure is used to induce functorial norms on the vector spaces $\mathbf{E}(\Sigma)$. A way out of these assumptions will be discussed in another publication.

Tracing

If $\mathcal{Z}$ is a stable unitary modular functor, we can apply the previous construction to its endomorphism theory. The trace gives a natural transformation of target theories

$$\eta_{\mathbf{\Sigma}} : \mathcal{C}^0(\mathcal{T}_\Sigma, \mathrm{End}\,\mathcal{Z}(\mathbf{\Sigma})) \Rightarrow \mathcal{C}^0(\mathcal{T}_\Sigma) \qquad \eta_{\mathbf{\Sigma}}(f)(\sigma) = \mathrm{Tr}\, f(\sigma)$$

In our vocabulary, a unitary modular functor has finite-dimensional spaces $\mathcal{Z}(\mathbf{\Sigma})$, so the trace is well-defined.

14 Geometric structures on moduli spaces of flat connections

14.1 Moduli space of flat connections

Throughout this section, G will be a simple, simply connected, compact Lie group. We denote $\mathscr{C}_G$ the space of conjugacy classes of G and $k_G : G \to \mathscr{C}_G$ the natural projection. If Σ is a bordered surface, we shall consider the moduli space of flat G-connections $\mathscr{M}_{G,\Sigma}$. We have the natural holonomy map around the boundary components

$$H_\partial : \mathscr{M}_{G,\Sigma} \longrightarrow \mathscr{C}_G^{\pi_0(\partial\Sigma)}.$$

If $h \in \mathscr{C}_G^{\pi_0(\partial\Sigma)}$, we denote

$$\mathscr{M}_{G,\Sigma}(h) = H_\partial^{-1}(h).$$

It is well-known that these spaces are stratified smooth spaces, where the stratification is specified by the centraliser of the holonomy group of a given flat connection. If now $\mathbf{\Sigma}$ is a pointed bordered surface, we can consider the based moduli space of flat G-connections $\widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}}$, meaning that we only mod out by the subgroup of the gauge group which is based at the marked points on the boundary. We then have a lift of the holonomy map

$$\tilde{H}_\partial : \widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}} \longrightarrow G^{\pi_0(\partial\Sigma)},$$

and of course a natural projection map

$$\tilde{\pi}_{\mathbf{\Sigma}} : \widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}} \longrightarrow \mathscr{M}_{G,\Sigma}$$

such that

$$H_\partial \circ \tilde{\pi}_{\mathbf{\Sigma}} = k_G \circ \tilde{H}_\partial.$$

14.2 Lattice gauge theory approach

Strategy

The moduli spaces $\mathscr{M}_{G,\Sigma}(h)$ for bordered surfaces admit the structure of a stratified symplectic space [47, 35]. The symplectic form on these spaces was first constructed by Narasimhan from a complex geometry point of view and then from an infinite dimensional symplectic reduction perspective by Atiyah and Bott [6] and finally from a representation variety point of view by Goldman [33, 34]. There is in fact a further fourth point of view due to Fock and Rosly [30], who construct a Poisson structure $\mathcal{P}$ on $\mathscr{M}_{G,\Sigma}$ whose symplectic leaves are exactly the $\mathscr{M}_{G,\Sigma}(h)$ with their Narasimhan-Atiyah-Bott-Goldman symplectic structures. The construction of Fock and Rosly requires the choice of an embedded fatgraph $\mathcal{G}$ in Σ, in such a way that Σ deformation retracts onto $\mathcal{G}$. If $V(\mathcal{G})$ and $E(\mathcal{G})$ are the set of vertices and edges of $\mathcal{G}$, then we have a homeomorphism

$$\mathscr{M}_{G,\Sigma} \cong G^{E(\mathcal{G})}/G^{V(\mathcal{G})}.$$

This setting directly adapts to pointed bordered surfaces $\mathbf{\Sigma}$. We introduce the set $\mathfrak{H}_{\mathbf{\Sigma}}$ of homotopy class relative to the boundary of fatgraphs $\mathcal{G}$ embedded in $\mathbf{\Sigma}$ such that each of the marked point on the boundary components of $\mathbf{\Sigma}$ is a univalent vertex of $\mathcal{G}$, all other vertices of $\mathcal{G}$ being at least trivalent, and $\mathbf{\Sigma}$ deformation retracts onto $\mathcal{G}$ (see Figure 21). Note that the definition forces $\Sigma \smallsetminus \mathcal{G}$ to be homeomorphic to a disjoint union of open disks, one for each boundary component of $\mathbf{\Sigma}$. In particular, those fatgraphs are different from the ones considered in Section 3.5. For each $\mathcal{G} \in \mathfrak{H}_{\mathbf{\Sigma}}$, we have homeomorphisms

$$\widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}} \cong G^{E(\mathcal{G})}/G^{V'(\mathcal{G})}. \tag{97}$$

where $V'(\mathcal{G})$ is the set of non-univalent vertices of $\mathcal{G}$. In particular, the holonomy around a boundary component b based at its marked point $o_b \in b$ can be directly read in this representation. Indeed, the based oriented loop going along $b \in \pi_0(\Sigma)$ following its orientation in $\partial\Sigma$ – that is, negative orientation if $b \in \partial_-\Sigma$, positive orientation if $b \in \partial_+\Sigma$ – deformation retracts onto a unique based loop on $\mathcal{G}$ which we decompose as a succession of oriented edges of $\mathcal{G}$. If $x \in G^{E(\mathcal{G})}/G^{V'(\mathcal{G})}$, $\tilde{H}_\partial(x)(b)$ is then the order-preserving product of the group elements $x(e)$ attached to each of the oriented edges e met, with the convention that the group elements on an oriented edge and on the oppositely oriented one are inverse of each other.

The advantage of considering this lattice gauge theory description of the space, is that even though $\widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}}$ is singular, we can simply consider various geometric structures on $\widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}}$ as the corresponding $G^{V'(\mathcal{G})}$-invariant geometric structures on $G^{E(\mathcal{G})}$. For a large class of geometric structures, it has already been proved that the notion is independent of the fatgraph $\mathcal{G}$. This is certainly the case for Poisson bivectors [30]. In particular we can define

$$\mathcal{C}^0\big(\widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}}, \Lambda^2 T_{\widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}}}\big) \coloneqq \sum_{\mathcal{G}\in\mathfrak{H}_{\mathbf{\Sigma}}} \mathcal{C}^0\big(G^{E(\mathcal{G})}, \Lambda^2 T_{G^{E(\mathcal{G})}}\big)^{G^{V'(\mathcal{G})}} \big/ \sim\,, \tag{98}$$

where the equivalence relation is induced by the natural identification of the spaces (97) for two different fatgraphs. In fact, (98) is merely a trick to avoid the specification of a fatgraph. In practice, one often chooses a fatgraph $\mathcal{G} \in \mathfrak{H}_{\mathbf{\Sigma}}$ and then work with

$$\mathcal{C}^0(\widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}}, \Lambda^2 T_{\widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}}}) \cong \mathcal{C}^0\big(G^{E(\mathcal{G})}, \Lambda^2 T_{G^{E(\mathcal{G})}}\big)^{G^{V'(\mathcal{G})}}\,.$$

Pre-target theory.

We will apply the idea of fibering (Section 10.3), and therefore start our description with a pre-target theory. For an object $\mathbf{\Sigma}$ in $\mathrm{Bord}_1^\bullet$ we let

$$\mathbf{F}(\Sigma) = \mathcal{C}^0(\widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}}, \Lambda^2 T_{\widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}}})\,.$$

It is equipped with a Banach norm induced by the supremum of the norm on $\Lambda^2 T$ induced by the G-bi-invariant metric on the tangent space, normalized so that G has volume 1. Here we use the assumption that G is compact. We remark that different choice of fatgraphs give the same norms.

The disjoint union morphisms is obtained as follows. We have the following natural projection maps of moduli spaces

$$p_i : \widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}_1\cup\mathbf{\Sigma}_2} \longrightarrow \widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}_i}\,.$$

Then there is a natural isomorphism

$$T_{\widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}_1\cup\mathbf{\Sigma}_2}} \cong p_1^* T_{\widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}_1}} \oplus p_2^* T_{\widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}_2}}.$$

Thus we can define

$$\sqcup : \mathbf{F}(\mathbf{\Sigma}_1) \times \mathbf{F}(\mathbf{\Sigma}_2) \longrightarrow \mathbf{F}(\mathbf{\Sigma}_1 \cup \mathbf{\Sigma}_2)$$

by

$$\Pi_1 \sqcup \Pi_2 = p_1^*(\Pi_1) + p_2^*(\Pi_2)\,.$$

We now describe the glueing morphisms. Let $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ be objects in $\mathrm{Bord}_1^\bullet$ and $\mathbf{\Sigma}$ be the object of $\mathrm{Bord}_1^\bullet$ obtained from glueing $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ along a subset $b \subset \pi_0(\partial\Sigma_1)\times\pi_0(\partial\Sigma_2)$. Given two fatgraphs $\mathcal{G}_i \in \mathfrak{H}_{\mathbf{\Sigma}_i}$, for each $\gamma = (\gamma_1,\gamma_2) \in b$, γ_i deformation retracts to an oriented path e_γ^1 on $\mathcal{G}_i$ for $i \in \{1,2\}$. We identify e_γ^1 and e_γ^2 with opposite orientation so as to obtain a well-defined homotopy class of fatgraphs $\mathcal{G} \in \mathfrak{H}_{\mathbf{\Sigma}}$. Let us denote V_i^b the set of edges of $\mathcal{G}_i$ which are incident to $\mathrm{pr}_i(b) \subset \pi_0(\partial\Sigma_i)$. We consider the smooth subvariety

$$(G^{E(\mathcal{G}_1)} \times G^{E(\mathcal{G}_2)})^= = \big\{(x_1,x_2) \in G^{E(\mathcal{G}_1)} \times G^{E(\mathcal{G}_2)}, \qquad x_1|_{V_1^b} = x_2|_{V_2^b}\big\}\,,$$

and we see that

$$\left(G^{E(\mathcal{G}_1)} \times G^{E(\mathcal{G}_2)}\right)^{=} \cong G^{E(\mathcal{G})} .$$

Besides, we have a diffeomorphism

$$G^{E(\mathcal{G})} \times G^{\pi_0(\gamma)} \cong G^{E(\mathcal{G}_1)} \times G^{E(\mathcal{G}_2)} , \tag{99}$$

which induces a smooth principal $G^{\pi_0(\gamma)}$-bundle

$$q_b : G^{E(\mathcal{G}_1)} \times G^{E(\mathcal{G}_2)} \longrightarrow G^{E(\mathcal{G})} ,$$

obtained by multiplying in G on all edges corresponding to the second factor in the left-hand side of (99). From this we see that there is a unique morphism

$$\mathbf{\Theta}_b : \mathbf{F}(\mathbf{\Sigma}_1) \times \mathbf{F}(\mathbf{\Sigma}_2) \longrightarrow \mathbf{F}(\mathbf{\Sigma})$$

such that, for any $f_i \in \mathcal{C}^0(G^{E(\mathcal{G}_i)})^{V'(\mathcal{G}_i)}$ we have

$$\mathbf{\Theta}_b(\Pi_1, \Pi_2)(f_1, f_2) = (\Pi_1 \sqcup \Pi_2)(q_b^* f_1, q_b^* f_2) .$$

Target theory

We now set

$$\mathbf{E}(\mathbf{\Sigma}) = \mathcal{C}^0(T_{\mathbf{\Sigma}} \times \widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}} , m_{G,\mathbf{\Sigma}}^* \Lambda^2 T_{\widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}}}) , \tag{100}$$

where

$$m_{G,\mathbf{\Sigma}} : \mathcal{T}_{\Sigma} \times \widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}} \longrightarrow \widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}}$$

is the projection onto the second factor. The structures we have described on $\mathbf{F}(\Sigma)$ make (100) a target theory using the remarks of Section 10.3.

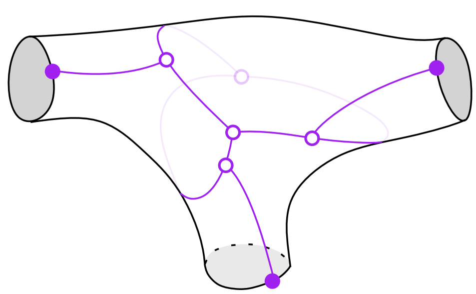

Figure 21: A uni-trivalent fatgraph in $\mathfrak{H}_{\mathbf{P}}$. Note that the procedure of glueing of such graphs will *a priori* produce vertices of valency higher than 3.

14.3 Application to Fock-Rosly Poisson structure

We shall now give an application of this setup. If $\mathbf{P}$ is a pair of pants and $\mathbf{T}$ a torus with one boundary component, we denote $\Pi_{\mathbf{P}} \in \mathbf{F}(\mathbf{P})$ and $\Pi_{\mathbf{T}} \in \mathbf{F}(\mathbf{T})$ the Fock-Rosly Poisson structures on $\widetilde{\mathscr{M}}_{G,\mathbf{P}}$ and $\widetilde{\mathscr{M}}_{G,\mathbf{T}}$ [30].

Theorem 14.1 *Recall the functions on Teichmüller spaces* $(\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D})$ *defining Mirzakhani initial data in Section 10.1. The initial data*

$$\begin{aligned} A_{\mathbf{P}} &= m^*_{G,\mathbf{P}}(\Pi_{\mathbf{P}})\,, \\ B^{b,b_1}_{\mathbf{P}} &= \mathsf{B}^{b,b_1}_{\mathbf{P}} \cdot m^*_{G,\mathbf{P}}(\Pi_{\mathbf{P}})\,, \\ C^{b_1}_{\mathbf{P}} &= \mathsf{C}^{b_1}_{\mathbf{P}} \cdot m^*_{G,\mathbf{P}}(\Pi_{\mathbf{P}})\,, \\ D_{\mathbf{T}} &= \mathsf{D}_{\mathbf{T}} \cdot m^*_{G,\mathbf{T}}(\Pi_{\mathbf{T}})\,, \end{aligned}$$

is M-admissible. For any object $\mathbf{\Sigma}$ *in* $\mathrm{Bord}^\bullet_1$*, the corresponding GR amplitude* $\Omega_{\mathbf{\Sigma}}$ *coincides with the Fock-Rosly Poisson structure on* $\widetilde{\mathscr{M}}_{G,\mathbf{\Sigma}}$*.*

Proof. This result follows directly from the Mirzakhani-McShane identities and the compatibility of the Fock-Rosly construction under glueing. ■

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